

ARITHMETIC VOLUME OF SHTUKAS AND LANGLANDS DUALITY

ZEYU WANG AND WENQING WEI

ABSTRACT. We extend the work of Feng–Yun–Zhang relating the arithmetic volume of Shtukas with derivatives of zeta functions by allowing arbitrary coweights for split semisimple algebraic groups. As in their original work, the formula involves some numbers called eigenweights. We obtain uniform formulas for the eigenweights in terms of the Langlands dual group, marking the first structural role for the dual group in such formulas governing derivatives of L -functions.

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1. INTRODUCTION

1.1. **Motivation.** Shimura varieties play a fundamental role in the Langlands correspondence over number fields, and their geometry encodes deep arithmetic information. One manifestation of this philosophy is that the volume of Shimura varieties is related to Dirichlet L -functions of number fields, while their arithmetic volume, namely the volume of integral models, is related to the first derivative of these L -functions. See [FYZ26, §1.2] for a summary of the literature.

The moduli of Shtukas provide the function field analogue of (integral models of) Shimura varieties. In [FYZ26], the arithmetic volume of moduli stacks of Shtukas with minuscule modification type was studied and related to higher derivatives of the zeta function of the curve. Compared with the number field setting, two striking new features arise. First, derivatives of arbitrary order naturally appear, in contrast to the number field case where only the first derivative is expected. Second, certain nontrivial constants, referred to as *eigenweights*, enter the formulas. While their meaning remains somewhat mysterious in the number field setting, they admit a natural geometric interpretation in the function field context. This additional structure can in turn be used to predict conjectural formulas for arithmetic volumes of Shimura varieties, providing part of the motivation for the work of [FYZ26].

However, the results of [FYZ26] are restricted to minuscule modification types, which obscures the conceptual unity of the theory. Moreover, although eigenweights admit a geometric definition, their explicit computation remains intricate. Even for groups of type A , the formulas obtained in [Fen26] involve complicated information such as the character tables of symmetric groups.

In this article, we extend these results to arbitrary modification types and provide a conceptual and uniform description of eigenweights in terms of the Langlands dual group. To the best of our knowledge, this is the first instance in which the Langlands dual group plays a direct and structural role in formulas governing derivatives of L -functions. In addition, we carry out explicit computations of eigenweights in most cases of fundamental importance, leading to formulas that are more elementary than those previously known.

1.2. **Main result.** We now present the main result of this article and recall the necessary background.

1.2.1. *Moduli stack of Shtukas.* Fix a proper smooth geometrically connected curve C defined over a finite field \mathbb{F}_q . Let G be a split semisimple algebraic group defined over \mathbb{F}_q , and let Bun_G be the moduli stack of principal G -bundles on C . Fix a maximal torus $T \subset G$. For each dominant coweight $\lambda \in X_*(T)_+$, one has the Hecke stack

$$\text{Bun}_G \xleftarrow{\overline{\overline{h}}} \text{Hk}_{G, \leq \lambda} \xrightarrow{\overline{\overline{h}}} \text{Bun}_G .$$

Here, we use $\overline{\overline{h}}$ (resp. $\overline{\overline{h}}$) to denote the map remembering only the G -bundle on the left (resp. right). We put an overline to distinguish it from the Hecke map that remembers also the point on the curve.

The *moduli stack of Shtukas with one leg* $\text{Sht}_{G, \leq \lambda}$ is defined via the Cartesian square

$$\begin{array}{ccc} \text{Sht}_{G, \leq \lambda} & \xrightarrow{f_{\text{Sht}}} & \text{Hk}_{G, \leq \lambda} \\ \downarrow & & \downarrow (\overline{\overline{h}}, \text{Frob} \circ \overline{\overline{h}}) \\ \text{Bun}_G & \xrightarrow{\Delta_{\text{Bun}_G}} & \text{Bun}_G \times \text{Bun}_G \end{array}$$

where $\text{Frob} : \text{Bun}_G \rightarrow \text{Bun}_G$ is the Frobenius map and $\Delta_{\text{Bun}_G} : \text{Bun}_G \rightarrow \text{Bun}_G \times \text{Bun}_G$ is the diagonal map.

Moreover, for a sequence of dominant coweights $\lambda_I = (\lambda_1, \dots, \lambda_r) \in X_*(T)_+^r$ where $I = \{1, \dots, r\}$, one can define the iterated Hecke stack

$$\text{Hk}_{G, \leq \lambda_I} := \text{Hk}_{G, \leq \lambda_1} \times_{\text{Bun}_G} \text{Hk}_{G, \leq \lambda_2} \times_{\text{Bun}_G} \cdots \times_{\text{Bun}_G} \text{Hk}_{G, \leq \lambda_r} .$$

It defines a correspondence

$$\text{Bun}_G \xleftarrow{\overline{\overline{h}}_I} \text{Hk}_{G, \leq \lambda_I} \xrightarrow{\overline{\overline{h}}_I} \text{Bun}_G .$$

One defines the *moduli stack of (iterated) Shtukas with r -legs* via the Cartesian square

$$\begin{array}{ccc} \text{Sht}_{G, \leq \lambda_I} & \xrightarrow{f_{\text{Sht}, I}} & \text{Hk}_{G, \leq \lambda_I} \\ \downarrow & & \downarrow (\overline{\overline{h}}_I, \text{Frob} \circ \overline{\overline{h}}_I) \\ \text{Bun}_G & \xrightarrow{\Delta_{\text{Bun}_G}} & \text{Bun}_G \times \text{Bun}_G \end{array}$$

1.2.2. *Arithmetic volume of the moduli of Shtukas.* Let $d = \dim \text{Sht}_{G, \leq \lambda_I}$. For a top-degree cohomology class $\alpha \in H^{2d}(\text{Sht}_{G, \leq \lambda_I})$, viewed as a ‘‘top form,’’ it is natural to consider the corresponding ‘‘volume’’ of $\text{Sht}_{G, \leq \lambda_I}$. However, this notion is not well-behaved, since $\text{Sht}_{G, \leq \lambda_I}$ is neither proper nor smooth. In fact, one has $H^{2d}(\text{Sht}_{G, \leq \lambda_I}) = 0$ whenever $r > 0$.

In [FYZ26], the authors introduced an *ad hoc* definition of this volume in the special case where α can be thought of as $f_{\text{Sht}, I}^* \alpha_0$ for some $\alpha_0 \in H^{2d}(\text{Hk}_{G, \leq \lambda_I})$. The idea originates from the Grothendieck–Lefschetz trace formula. More precisely, one considers the maps

$$H_c^*(\text{Bun}_G) \xrightarrow{(\overline{\overline{h}}_I)^*} IH_c^*(\text{Hk}_{G, \leq \lambda_I}) \xrightarrow{(\overline{\overline{h}}_I)_!} H_c^{*-2d}(\text{Bun}_G),$$

where $IH_c^*(\text{Hk}_{G, \leq \lambda_I})$ denotes the compactly supported intersection cohomology of $\text{Hk}_{G, \leq \lambda_I}$.¹ We refer to §4.1 for a precise definition of these maps.

Given the data above, one considers the operator

$$\Gamma_{c, \alpha_0} : H_c^*(\text{Bun}_G) \rightarrow H_c^*(\text{Bun}_G) \tag{1.1}$$

defined by

$$\Gamma_{c, \alpha_0}(-) := (\overline{\overline{h}}_I)_! ((\overline{\overline{h}}_I)^*(-) \cup \alpha_0).$$

Note that the operator Γ_{c, α_0} preserves the cohomological degree. One defines the *arithmetic volume* of the moduli space $\text{Sht}_{G, \leq \lambda_I}$ with respect to ‘‘ α ’’ to be the number

$$\text{vol}(\text{Sht}_{G, \leq \lambda_I}, \alpha) := \text{tr}(\text{Frob} \circ \Gamma_{c, \alpha_0}, H_c^*(\text{Bun}_G)). \tag{1.2}$$

See Remark 2.10 for a discussion of convergence issues related to this definition.

¹In [FYZ26], only the case where $\text{Hk}_{G, \leq \lambda_I}$ is smooth is considered. The extension to the non-smooth case via intersection cohomology is straightforward.

Example 1.1. When $r = 0$, one has $\mathrm{Hk}_{G,\emptyset} = \mathrm{Bun}_G$ and $\mathrm{Sht}_{G,\emptyset}^0 = \mathrm{Bun}_G(\mathbb{F}_q)$. We choose $\alpha_0 = 1 \in H^0(\mathrm{Bun}_G)$. In this case, we have $\mathrm{vol}(\mathrm{Bun}_G(\mathbb{F}_q), 1) = \mathrm{tr}(\mathrm{Frob}, H_c^*(\mathrm{Bun}_G))$. Modulo convergence issues, the Grothendieck–Lefschetz fixed point formula predicts that $\mathrm{vol}(\mathrm{Bun}_G(\mathbb{F}_q), 1) = |\mathrm{Bun}_G(\mathbb{F}_q)|$. This matches the natural expectation that the volume of the discrete stack $\mathrm{Bun}_G(\mathbb{F}_q)$ is its size.

1.2.3. *Main result: same modification type.* Now we specialize to a very canonical choice of α_0 . When $r = 1$, there is a distinguished line bundle $\mathcal{L}_{\mathrm{det}} \in \mathrm{Pic}(\mathrm{Hk}_G)_{\mathbb{Q}}$ called the *determinant line bundle*, obtained by pulling back its local counterpart defined in §4.1.

For general r , consider the natural projections $r_i : \mathrm{Hk}_{G,\leq\lambda_I} \rightarrow \mathrm{Hk}_{G,\leq\lambda_i}$ for $i \in I$. Define

$$\mathcal{L}_{\mathrm{det},I} := ((r_i)^* \mathcal{L}_{\mathrm{det}})_{i \in I} \in \mathrm{Pic}(\mathrm{Hk}_{G,\leq\lambda_I})_{\mathbb{Q}}^I.$$

We take

$$\alpha_0 = \prod_{i=1}^r c_1((r_i)^* \mathcal{L}_{\mathrm{det}})^{d_i} \in H^{2d}(\mathrm{Hk}_{G,\leq\lambda_I}^1), \quad \text{where } d_i = \langle 2\rho, \lambda_i \rangle + 1.$$

In this case, we write

$$\Gamma_{c,\mathcal{L}_{\mathrm{det},I}} = \Gamma_{c,\alpha_0} \tag{1.3}$$

$$\mathrm{vol}(\mathrm{Sht}_{G,\leq\lambda_I}, f_{\mathrm{Sht},I}^* \mathcal{L}_{\mathrm{det},I}) = \mathrm{vol}(\mathrm{Sht}_{G,\leq\lambda_I}, \alpha).$$

We first state a version of the main result when all $\{\lambda_i, 1 \leq i \leq r\}$ are the same, which already reflects the most interesting aspects of this problem. See §1.2.4 for a more general version.

Theorem 1.2. For $\lambda_I = (\lambda, \lambda, \dots, \lambda)$ where $\lambda \in X_*(T)_+$, assuming $\mathrm{Sht}_{G,\lambda_I} \neq \emptyset$, we have

$$\mathrm{vol}(\mathrm{Sht}_{G,\leq\lambda_I}, f_{\mathrm{Sht},I}^* \mathcal{L}_{\mathrm{det},I}) = |\pi_1(G)| q^{\dim \mathrm{Bun}_G} (\log q)^{-r} \left(\frac{d}{ds} \right) \Big|_{s=0} \left(q^{(2g-2)b_\lambda s} \prod_{i=1}^n \zeta_C(-\epsilon_{\lambda,i}s + d_i) \right). \tag{1.4}$$

Here,

- $n = \mathrm{rk}(G)$ is the rank of the group G ,
- $d_i, 1 \leq i \leq n$ are the degrees of fundamental invariants of G , that is, the degree of generators of the free polynomial ring $\overline{\mathbb{Q}}_\ell[\mathfrak{g}]^G$ where \mathfrak{g} is the Lie algebra of G .
- $b_\lambda \in \mathbb{Q}, \epsilon_{\lambda,i} \in \overline{\mathbb{Q}}$ are some constants,
- $\zeta_C(s)$ is the zeta function of C .
- g is the genus of the curve C .

When λ is minuscule, Theorem 1.2 is a special case of [FYZ26, Theorem 1.3.8].²

The numbers $\epsilon_{\lambda,i}$ are called *eigenweights* in [FYZ26]. They are rational numbers when G is not of type D_n for $n \geq 4$ even. See Example 2.8 for the precise meaning of these numbers. In §5.1, we explicitly calculate these numbers for all coweights of classical groups and most fundamental coweights for exceptional groups, resulting in formulas that are more elementary than those in [Fen26].

The constants b_λ are less emphasized in [FYZ26] as they are easy when λ is minuscule. See §5.2 for some computational results on these numbers.

In §4, we will give a uniform description of the constants $b_\lambda, \epsilon_{\lambda,i}$ in terms of the Langlands dual group \check{G} of G .

Remark 1.3. Generalization of Theorem 1.2 to the case that G is split reductive should be straightforward. We only state the result for semisimple groups to keep the formula clean and compact.

Remark 1.4. Theorem 1.2 can be regarded as an instance of relative Langlands duality in the sense of [BZSV24]: On the automorphic side, one takes the trivial spherical G -variety $G \curvearrowright$ point corresponding to the constant period; on the spectral side, one takes the twisted cotangent bundle $\check{G} \curvearrowright T_{\check{q}}^*(\check{G}/\check{N})$ corresponding to the Whittaker period. From this point of view, the constants $b_\lambda, \epsilon_{\lambda,i}$ can be read off from the Poisson structure on the local Plancherel algebra introduced in [BZSV24, §8] (see Remark 4.5).

²Strictly speaking, the line bundle used in [FYZ26, Theorem 1.3.8] is different, but it is easy to compare the two results.

1.2.4. *Main result: different modification types.* Now we give a more general version of Theorem 1.2 in the case where $\lambda_I = (\lambda_1, \dots, \lambda_r) \in X_*(T)_+^r$ and the λ_i are not necessarily equal.

For each $i \in I$, define a differential operator on $\mathbb{R}^n = \{(s_1, \dots, s_n) \mid s_j \in \mathbb{R}\}$ by

$$D_{\lambda_i} := (2g - 2)b_{\lambda_i} - (\log q)^{-1} \sum_{j=1}^n \epsilon_{\lambda_i, j} \partial_{s_j}. \quad (1.5)$$

Here the numbers b_{λ_i} and $\epsilon_{\lambda_i, j}$ are the same as those in Theorem 1.2 for the coweight λ_i . See §2.4.2 for the ordering of these numbers.

Consider the L -function

$$\mathcal{L}_{C, G}(s_1, \dots, s_n) := \prod_{i=1}^n \zeta_C(s_i + d_i). \quad (1.6)$$

The following theorem is a special case of Theorem 2.13.

Theorem 1.5. *Under Assumption 2.12 (which always holds when G is not of type D_n for n even) and assuming $\text{Sht}_{G, \leq \lambda_I} \neq \emptyset$, we have*

$$\text{vol}(\text{Sht}_{G, \leq \lambda_I}, f_{\text{Sht}, I}^* \mathcal{L}_{\det, I}) = |\pi_1(G)| q^{\dim \text{Bun}_G} \left(\left(\prod_{i=1}^r D_{\lambda_i} \right) \mathcal{L}_{C, G}(s_1, \dots, s_n) \right) \Big|_{s_1 = \dots = s_n = 0}. \quad (1.7)$$

1.3. **New ingredients.** Theorem 1.5 generalizes [FYZ26, Theorem 1.3.8] beyond the minuscule case. As is clear from the formulation, the main task in proving the theorem is to compute the operator $\Gamma_{c, \mathcal{L}_{\det, I}}$ in (1.3), which we call *(global) relative Hecke operators*. In [FYZ26], the computation of relative Hecke operators relies on the Vinberg semigroup and the wonderful compactification of the adjoint group, which are only applicable in the minuscule case.

Instead, we develop a new approach to compute relative Hecke operators: we study their local counterparts (2.13), which we call *local relative Hecke operators*, and then compute the global ones via a local–global compatibility result (Theorem 3.2). This approach has several advantages. First, it allows us to handle the non-minuscule case, and indeed all modification types in a uniform way. Second, our local models are directly related to the Satake category, which enables a uniform description of the eigenweights in terms of the Langlands dual group (achieved in §4). Finally, our approach is closely connected to the relative Langlands duality in the sense of [BZSV24], and is directly comparable with the approach in [LW25] which deals with the strongly tempered case. This perspective suggests a unified conceptual framework for understanding Gross–Zagier type formulas over function fields, namely, formulas in which higher derivatives of L -functions arise. See Remark 4.5 for further discussion of this connection.

1.4. **Notations.** We use C to denote a smooth geometrically connected projective curve defined over \mathbb{F}_q . Let $\xi \in H^2(C)$ be the fundamental class of C . Let $D = \text{Spec } \mathbb{F}_q[[t]]$ be the formal disc, and $\text{Aut}(D)$ be the group of origin fixing automorphisms of D .

Throughout the paper, we work with a split semisimple connected algebraic group G . Fix a maximal torus and Borel subgroup $T \subset B \subset G$. We use W to denote the Weyl group of G .

We use L^+G, LG to denote the jet, loop group of G . We use $\text{Gr}_G := LG/L^+G$ to denote the affine Grassmannian, which is an ind-scheme. We define the local Hecke stack $\text{Hk}_G^1 := (L^+G \backslash \text{Gr}_G) / \text{Aut}(D)$. As a global counterpart, we have the global Hecke stack Hk_G which admits a map $\vec{h} : \text{Hk}_G \rightarrow \text{Bun}_G \times C$ with each fiber isomorphic to Gr_G .

For each $\lambda \in X_*(T)$, we have the closed Schubert cells $\text{Gr}_{G, \leq \lambda} \subset \text{Gr}_G$, $\text{Hk}_{G, \leq \lambda}^1 \subset \text{Hk}_G^1$, $\text{Hk}_{G, \leq \lambda} \subset \text{Hk}_G$ defined such that $\text{Gr}_{G, \leq \lambda} = \text{Gr}_{G, \leq \lambda^+}$ where $\lambda^+ \in X_*(T)_+$ is the unique dominant element in $W \cdot \lambda \subset X_*(T)$.

Define $k = \overline{\mathbb{Q}}_\ell$. For a prestack X , we use $\text{Shv}(X)$ to denote the category of (ind-)constructible étale sheaves on X with coefficient in k . We refer to [LW25, §4.1] for a detailed definition of this category. We use $k_X \in \text{Shv}(X)$ to denote the constant sheaf on X , and $\omega_X = (X \rightarrow *)^! k \in \text{Shv}(X)$ to denote the dualizing sheaf on X whenever the $!$ -pullback functor is defined. For a prestack X , we have its cochain complex $\Gamma(X) := \Gamma_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, k)$. Define the cohomology ring $H^*(X) := \bigoplus_{d \in \mathbb{Z}} H^d \Gamma(X)$. We use $\langle n \rangle = [n](n/2)$ to denote the Tate twist by $n/2$ and cohomological shift by n .

Although this is not essential for our purposes, we work by default with $(\infty, 1)$ -categories. For a category \mathcal{C} and objects $x, y \in \mathcal{C}$, we denote by $\text{Hom}(x, y)$ the mapping space from x to y , and write $\text{Hom}^0(x, y) := \pi_0(\text{Hom}(x, y))$.

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2. GENERAL FORMALISM

In this section, we set up a general framework for defining and computing the operator in (1.3), which we call *(global) relative Hecke operators*. Namely, whenever the operator has a local origin (coming from a local volume datum as defined in Definition 2.1), we are able to compute it. The main result of this section is Theorem 2.13, which generalizes Theorem 1.2 and Theorem 1.5.

We refer to §1.4 for basic notations.

2.1. Volume data. We first introduce the general setup that we can define and compute the relative Hecke operators. To summarize, whenever one has some local volume data (to be introduced in §2.1.1), we will be able to define some relative Hecke operators (to be defined in §2.2.1).

2.1.1. Local volume data. The local Hecke stack Hk_G^1 introduced in §1.4 fits into a correspondence

$$\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)) \xleftarrow{\overleftarrow{h}_1} \mathrm{Hk}_G^1 \xrightarrow{\overrightarrow{h}_1} \mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)) \quad (2.1)$$

where the two maps have clear meaning from the description $\mathrm{Hk}_G^1 = (L^+G \rtimes \mathrm{Aut}(D)) \backslash (LG \rtimes \mathrm{Aut}(D)) / (L^+G \rtimes \mathrm{Aut}(D))$.

Definition 2.1. We define a *local volume datum* to be a triple $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ in which:

- $\mathcal{K} \in \mathrm{Shv}(\mathrm{Hk}_G^1)$ is a sheaf supported on $\mathrm{Hk}_{G, \leq \lambda}^1 \subset \mathrm{Hk}_G^1$ for some $\lambda \in X_*(T)_+$.
- $\mathfrak{c}^1 \in \mathrm{Corr}_{\mathrm{Hk}_G^1, \mathcal{K} \langle -d_c \rangle}(\underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))}, \underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))})$ is a *cohomological correspondence*.³ Here,

$$\mathrm{Corr}_{\mathrm{Hk}_G^1, \mathcal{K} \langle -d_c \rangle}(\underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))}, \underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))}) = \mathrm{Hom}^0(\overleftarrow{h}_{1,!} \mathcal{K} \langle -d_c \rangle, \underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))}). \quad (2.2)$$

- $\mathfrak{d}^1 \in \mathrm{Hom}^0(\underline{k}_{\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))}, \overrightarrow{h}_{1,!} \mathcal{K} \langle d_\mathfrak{d} \rangle)$.⁴

Here, $d_c, d_\mathfrak{d} \in \mathbb{Z}$ are some arbitrary integers, which are part of the datum. Define the *degree* of the local volume datum to be the integer $d_v = d_c + d_\mathfrak{d}$.

Example 2.2 (Local volume datum for determinant line bundle). For each dominant coweight $\lambda \in X_*(T)_+$, we have the local volume datum $v_{\lambda, \det} = (\mathrm{IC}_\lambda, \mathfrak{c}_{\lambda, \det}^1, \mathfrak{d}_\lambda^1)$ (see Definition 4.1).

Example 2.3 (Cup product with a cohomology class). For each local volume datum $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ and a cohomology class $\alpha \in H^{|\alpha|}(\mathrm{Hk}_G^1)$, one can define a new local volume datum $\alpha \cdot v := (\mathcal{K}, \alpha \cup \mathfrak{c}^1, \mathfrak{d}^1)$ which has degree $d_v + |\alpha|$.

2.1.2. Global volume data. We now introduce global volume data.

Definition 2.4. We define a *global volume datum* to be a triple $v = (\mathcal{K}, \mathfrak{c}, \mathfrak{d})$ in which:

- $\mathcal{K} \in \mathrm{Shv}(\mathrm{Hk}_G)$ is a sheaf supported on $\mathrm{Hk}_{G, \leq \lambda} \subset \mathrm{Hk}_G$ for some $\lambda \in X_*(T)_+$.
- $\mathfrak{c} \in \mathrm{Corr}_{\mathrm{Hk}_G, \mathcal{K} \langle -d_c \rangle}(\underline{k}_{\mathrm{Bun}_G \times C}, \underline{k}_{\mathrm{Bun}_G \times C})$ is a cohomological correspondence. Here,

$$\mathrm{Corr}_{\mathrm{Hk}_G, \mathcal{K} \langle -d_c \rangle}(\underline{k}_{\mathrm{Bun}_G \times C}, \underline{k}_{\mathrm{Bun}_G \times C}) = \mathrm{Hom}^0(\overleftarrow{h}_! \mathcal{K} \langle -d_c \rangle, \underline{k}_{\mathrm{Bun}_G \times C}). \quad (2.3)$$

- $\mathfrak{d} \in \mathrm{Hom}^0(\underline{k}_{\mathrm{Bun}_G \times C}, \overrightarrow{h}_! \mathcal{K} \langle d_\mathfrak{d} \rangle)$.

Here, $d_c, d_\mathfrak{d} \in \mathbb{Z}$ are some arbitrary integers, which are part of the datum. Define the degree of the global volume datum to be the integer $d_v = d_c + d_\mathfrak{d}$. Here, the maps $\overleftarrow{h}, \overrightarrow{h}$ are the top maps in (2.4).

³We refer to [LW25, §4.2] for a general treatment of cohomological correspondences with kernels. Here, one can forget about the interpretation as a cohomological correspondence and directly work with the Hom space.

⁴The element \mathfrak{d}^1 can be regarded as a cohomological correspondence with kernel the Verdier dual of \mathcal{K} .

2.1.3. *Local-to-global construction.* From a local volume datum $(\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$, we can construct a global volume datum $f^*v = (f_{\text{Hk}}^* \mathcal{K}, \mathfrak{c}, \mathfrak{d})$ as follows: Consider the diagram

$$\begin{array}{ccccc} \text{Bun}_G \times C & \xleftarrow{\overleftarrow{h}} & \text{Hk}_G & \xrightarrow{\overrightarrow{h}} & \text{Bun}_G \times C \\ \downarrow f & & \downarrow f_{\text{Hk}} & & \downarrow f \\ \mathbb{B}(L^+G \rtimes \text{Aut}(D)) & \xleftarrow{\overleftarrow{h}_1} & \text{Hk}_G^1 & \xrightarrow{\overrightarrow{h}_1} & \mathbb{B}(L^+G \rtimes \text{Aut}(D)) \end{array} \quad (2.4)$$

in which the vertical maps are given by restriction to the formal disc near the point on the curve C . The top horizontal row is the global counterpart of the correspondence (2.1). Both squares in (2.4) are Cartesian.

Define

$$\mathfrak{c} := f^* \mathfrak{c}^1 \in \text{Corr}_{\text{Hk}_G, f_{\text{Hk}}^*} \mathcal{K} \langle -d_{\mathfrak{c}} \rangle (\underline{k}_{\text{Bun}_G \times C}, \underline{k}_{\text{Bun}_G \times C}) = \text{Hom}^0(\overleftarrow{h}_! f_{\text{Hk}}^* \mathcal{K} \langle -d_{\mathfrak{c}} \rangle, \underline{k}_{\text{Bun}_G \times C}) \quad (2.5)$$

as the composition

$$\overleftarrow{h}_! f_{\text{Hk}}^* \mathcal{K} \langle -d_{\mathfrak{c}} \rangle \cong f^* \overleftarrow{h}_{1,!} \mathcal{K} \langle -d_{\mathfrak{c}} \rangle \xrightarrow{f^* \mathfrak{c}^1} f^* \underline{k}_{\mathbb{B}(L^+G \rtimes \text{Aut}(D))} \cong \underline{k}_{\text{Bun}_G \times C},$$

and

$$\mathfrak{d} := f^* \mathfrak{d}^1 \in \text{Hom}^0(\underline{k}_{\text{Bun}_G \times C}, \overrightarrow{h}_! f_{\text{Hk}}^* \mathcal{K} \langle d_{\mathfrak{d}} \rangle) \quad (2.6)$$

as the composition

$$\underline{k}_{\text{Bun}_G \times C} \cong f^* \underline{k}_{\mathbb{B}(L^+G \rtimes \text{Aut}(D))} \xrightarrow{f^* \mathfrak{d}^1} f^* \overrightarrow{h}_{1,!} \langle d_{\mathfrak{d}} \rangle \cong \overrightarrow{h}_! f_{\text{Hk}}^* \mathcal{K} \langle d_{\mathfrak{d}} \rangle.$$

Throughout the article, we will always work with global volume data coming from local volume data as above.

2.2. Relative Hecke operators. In this section, we will introduce relative Hecke operators associated with volume data in both local and global settings.

- In §2.2.1, we introduce global relative Hecke operators, which will exactly generalize the operators in (1.3).
- In §2.2.2, we introduce local relative Hecke operators, which serve as local models for the global ones.

2.2.1. *Global relative Hecke operators.* Now we introduce global relative Hecke operators associated to a global volume datum $v = (\mathcal{K}, \mathfrak{c}, \mathfrak{d})$.

We have a map

$$\Gamma_{\mathfrak{c}, v}^C : H_c^*(\text{Bun}_G \times C) \xrightarrow{H_c^*(\mathfrak{d})} H_c^{*+d_{\mathfrak{d}}}(\text{Hk}_G, \mathcal{K}) \xrightarrow{H_c^*(\mathfrak{c})} H_c^{*+d_v}(\text{Bun}_G \times C). \quad (2.7)$$

Since the (graded) dual of $H_c^*(\text{Bun}_G \times C)$ is $H^*(\text{Bun}_G \times C)(\dim \text{Bun}_G + 1)$. Ignoring the Tate twist, the (graded) dual of (2.7) is

$$\Gamma_v^C : H^*(\text{Bun}_G \times C) \xrightarrow{H^*(\mathfrak{c})} H_{-(**+d_{\mathfrak{c}})}^{BM}(\text{Hk}_G / \text{Bun}_G \times C, \mathcal{K}) \xrightarrow{H^*(\mathfrak{d})} H^{*+d_v}(\text{Bun}_G \times C) \quad (2.8)$$

in which $H_{-*}^{BM}(\text{Hk}_G / \text{Bun}_G \times C, \mathcal{K}) = \text{Hom}^*(\mathcal{K}, \omega_{\text{Hk}_G / \text{Bun}_G \times C})$. Here, we are using

$$\omega_{\text{Hk}_G / \text{Bun}_G \times C} = \overleftarrow{h}_! \underline{k}_{\text{Bun}_G \times C} \cong \overrightarrow{h}_! \underline{k}_{\text{Bun}_G \times C}.$$

The maps in (2.8) are defined by $H^*(\mathfrak{c}) = H_c^*(\mathfrak{c})^*$, $H^*(\mathfrak{d}) = H_c^*(\mathfrak{d})^*$. More explicitly, the first map can be identified with the map

$$\begin{aligned} H^*(\mathfrak{c}) : H^*(\text{Bun}_G \times C) &= \text{Hom}^*(\underline{k}_{\text{Bun}_G \times C}, \underline{k}_{\text{Bun}_G \times C}) \\ &\rightarrow \text{Hom}^*(\overleftarrow{h}_! \mathcal{K} \langle -d_{\mathfrak{c}} \rangle, \underline{k}_{\text{Bun}_G \times C}) \\ &\cong \text{Hom}^*(\mathcal{K} \langle -d_{\mathfrak{c}} \rangle, \overleftarrow{h}_! \underline{k}_{\text{Bun}_G \times C}) \\ &= H_{-(**+d_{\mathfrak{c}})}^{BM}(\text{Hk}_G / \text{Bun}_G \times C, \mathcal{K}) \end{aligned}$$

The second map can be identified with

$$\begin{aligned} H^*(\mathfrak{d}) : H_{-*}^{BM}(\mathrm{Hk}_G / \mathrm{Bun}_G \times C, \mathcal{K}) &\cong \mathrm{Hom}^*(\mathcal{K}, \overrightarrow{h}^1 \underline{k}_{\mathrm{Bun}_G \times C}) \\ &\cong \mathrm{Hom}^*(\overrightarrow{h}^1 \mathcal{K}, \underline{k}_{\mathrm{Bun}_G \times C}) \\ &\rightarrow \mathrm{Hom}^*(\underline{k}_{\mathrm{Bun}_G \times C}(-d_{\mathfrak{d}}), \underline{k}_{\mathrm{Bun}_G \times C}) \\ &= H^{*+d_{\mathfrak{d}}}(\mathrm{Bun}_G \times C) \end{aligned}$$

The maps $H^*(\mathfrak{c}), H^*(\mathfrak{d})$ are linear over $H^*(\mathrm{Bun}_G \times C)$. That is, for any $\alpha \in H^*(\mathrm{Bun}_G \times C)$, we have

$$H^*(\mathfrak{c})(\alpha \cdot -) = (\overleftarrow{h}^* \alpha) \cdot - \quad (2.9)$$

and

$$H^*(\mathfrak{d})(\overrightarrow{h}^* \alpha \cdot -) = \alpha \cdot -. \quad (2.10)$$

We further define the map

$$\Gamma_{c,v} : H_c^*(\mathrm{Bun}_G) \xrightarrow{\mathrm{pr}_1^*} H_c^*(\mathrm{Bun}_G \times C) \xrightarrow{\Gamma_{c,v}^C} H_c^{*+d_v}(\mathrm{Bun}_G \times C) \xrightarrow{\mathrm{pr}_{1,!}} H_c^{*+d_v-2}(\mathrm{Bun}_G) \quad (2.11)$$

and its dual

$$\Gamma_v : H^*(\mathrm{Bun}_G) \xrightarrow{\mathrm{pr}_1^*} H^*(\mathrm{Bun}_G \times C) \xrightarrow{\Gamma_v^C} H^{*+d_v}(\mathrm{Bun}_G \times C) \xrightarrow{\mathrm{pr}_{1,!}} H^{*+d_v-2}(\mathrm{Bun}_G). \quad (2.12)$$

We call operators defined in (2.7)(2.8)(2.11)(2.12) *global relative Hecke operators*.

Example 2.5. When $v = f^* v_{\lambda, \det}$ where $v_{\lambda, \det}$ is in Example 2.2, we have $\Gamma_{c, f^* v_{\lambda, \det}} = \Gamma_{c, \mathcal{L}_{\det, I}}$ where the later is defined in (1.3) in which one takes $r = 1$ and $\lambda_I = (\lambda)$.

2.2.2. Local relative Hecke operators. Now we introduce local relative Hecke operators associated to a local volume datum $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ as defined in Definition 2.1. They will play an important role in the computation of global relative Hecke operators and in formulating the main theorem.

Note that (2.8) has a local counterpart

$$\Gamma_v^1 : H^*(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))) \xrightarrow{H^*(\mathfrak{c}^1)} H_{-(**+d_c)}^{BM}(\mathrm{Hk}_G^1 / \mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)), \mathcal{K}) \xrightarrow{H^*(\mathfrak{d}^1)} H^{*+d_v}(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))). \quad (2.13)$$

Here, the maps $H^*(\mathfrak{c}^1)$ and $H^*(\mathfrak{d}^1)$ are linear over $H^*(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)))$. We call operators Γ_v^1 defined in (2.13) *local relative Hecke operators*.

2.3. Calculating local relative Hecke operators. Now we study local relative Hecke operators in more detail.

2.3.1. Cohomology of $\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))$. Let $R = H^*(\mathbb{B}T)$. Then $H^*(\mathbb{B}G) = R^W$ and $H^*(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))) = R^W[\hbar]$, where $\hbar := c_1(T_D) \in H^2([D / \mathrm{Aut}(D)]) \cong H^2(\mathbb{B} \mathrm{Aut}(D))$. Here T_D is the tangent bundle of D .

Note that $H^*(\mathbb{B}G)$ has a natural augmentation ideal $H^{>0}(\mathbb{B}G) \subset H^*(\mathbb{B}G)$. This defines a decreasing *augmentation filtration* $\{F_{\mathrm{aug}}^{\blacktriangleright} H^*(\mathbb{B}G)\}_{\blacktriangleright \in \mathbb{Z}_{\geq 0}}$ on $H^*(\mathbb{B}G)$ such that

$$F_{\mathrm{aug}}^i H^*(\mathbb{B}G) := (H^{>0}(\mathbb{B}G))^i \subset H^*(\mathbb{B}G).$$

Define the *Gross motive*

$$\mathbb{V} := \mathrm{Gr}_{\mathrm{aug}}^1 H^*(\mathbb{B}G). \quad (2.14)$$

We have a canonical isomorphism $\mathrm{Gr}_{\mathrm{aug}}^{\blacktriangleright} H^*(\mathbb{B}G) \cong \mathrm{Sym}^{\blacktriangleright}(\mathbb{V})$.

2.3.2. *Local relative Hecke operator on associated graded.* We keep working with a local volume datum $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ and assume $d_v = 2$.

Consider maps

$$\overleftarrow{h}_1^*, \overrightarrow{h}_1^* : H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \rightarrow H^*(\text{Hk}_G^1)$$

where $\overleftarrow{h}_1, \overrightarrow{h}_1$ are the bottom maps in (2.4). For $\alpha \in H^{|\alpha|}(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, define

$$[\alpha] = \hbar^{-1} \cdot (\overleftarrow{h}_1^* \alpha - \overrightarrow{h}_1^* \alpha) \in H^{|\alpha|-2}(\text{Hk}_G^1). \quad (2.15)$$

Here, we are using the fact that $H^*(\text{Hk}_G^1)$ is a flat $k[\hbar]$ -module.

Considering

$$\widetilde{\nabla}_v^1 := \Gamma_{[-1] \cdot v}^1(1) \in \text{End}(H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))) \quad (2.16)$$

where the dot product in $[-1] \cdot v$ is understood as in Example 2.3. Since $d_v = 2$, we know $\Gamma_v^1(1) \in H^2(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) = k \cdot \hbar$. Write

$$\Gamma_v^1(1) = -b_v \hbar \quad (2.17)$$

for $b_v \in k$.

For any $\alpha \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, note that

$$\begin{aligned} \Gamma_v^1(\alpha) &= H^*(\mathfrak{d}^1) \circ H^*(\mathfrak{c}^1)(\alpha) \\ &= H^*(\mathfrak{d}^1)((\overleftarrow{h}_1^*)^*(\alpha) H^*(\mathfrak{c}^1)1) \\ &= H^*(\mathfrak{d}^1)((\overrightarrow{h}_1^*)^*(\alpha) + \hbar[\alpha] H^*(\mathfrak{c}^1)1). \\ &= \alpha \Gamma_v^1(1) + \hbar \cdot \Gamma_{[\alpha] \cdot v}^1(1) \\ &= -b_v \hbar \cdot \alpha + \hbar \cdot \widetilde{\nabla}_v^1(\alpha) \end{aligned} \quad (2.18)$$

Define $\nabla_v^1 := \widetilde{\nabla}_v^1 \otimes_{k[\hbar]} k \in \text{End}(H^*(\mathbb{B}G))$. We have the following important observation:

Lemma 2.6. *The map $\nabla_v^1 : H^*(\mathbb{B}G) \rightarrow H^*(\mathbb{B}G)$ is a derivation. Moreover, it preserves the natural augmentation filtration $F_{\text{aug}}^\blacktriangleright H^*(\mathbb{B}G)$ defined in §2.3.1.*

Proof. Both follow from the computation that

$$\widetilde{\nabla}_v^1(\alpha\beta) = \alpha \widetilde{\nabla}_v^1(\beta) + \widetilde{\nabla}_v^1(\alpha)\beta + \hbar \Gamma_{[\alpha][\beta] \cdot v}^1(1) \quad (2.19)$$

for any $\alpha, \beta \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$. \square

Given Lemma 2.6, it is natural to consider

$$E_v := \text{Gr}_{\text{aug}}^1 \nabla_v^1 : \mathbb{V} \rightarrow \mathbb{V}. \quad (2.20)$$

We would like to call this the *relative Hecke operator on Gross motive*. Its eigenvalues form a (multi-)set

$$\{\epsilon_{v,i} \mid 1 \leq i \leq \dim \mathbb{V}\} \quad (2.21)$$

and are called *eigenweights* of v . See §2.4.2 for an ordering of the eigenweights when G is almost simple.

We have the following proposition to be compared with its global counterpart, Proposition 3.7:

Proposition 2.7. *The induced map $\overline{\Gamma}_v^1 := (\hbar^{-1} \Gamma_v^1) \otimes_{k[\hbar]} k$ preserves the filtration $F_{\text{aug}}^\blacktriangleright H^*(\mathbb{B}G)$. Under the canonical isomorphism $H^*(\mathbb{B}G) \cong \text{Sym}^\blacktriangleright(\mathbb{V})$, we have $\text{Gr}_{\text{aug}}^\blacktriangleright \overline{\Gamma}_v^1 = -b_v \cdot \text{id} + \nabla_{E_v} \in \text{End}(\text{Sym}^\blacktriangleright(\mathbb{V}))$. Here ∇_{E_v} is the derivation on $\text{Sym}^\blacktriangleright(\mathbb{V})$ such that $\nabla_{E_v}|_{\mathbb{V}} = E_v$.*

Example 2.8. When $v = v_{\lambda, \det}$ as in Example 2.2, we have $\epsilon_{\lambda,i} = \epsilon_{v_{\lambda, \det}, i}$ and $b_\lambda = b_{v_{\lambda, \det}}$. This gives the numbers involved in Theorem 1.2.

2.4. Volume of Shtukas: a general formulation. Now we formulate a general version of Theorem 1.5.

2.4.1. *Definition of volume.* For each local volume datum $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ as in Definition 2.1 and $e \in \pi_1(G) \cong \pi_0(\text{Bun}_G)$, assuming $\mathcal{K} \in \text{Shv}(\text{Hk}_G^{1,e\mathcal{K}})$ for some $e_{\mathcal{K}} \in \pi_1(G)$,⁵ we consider the corresponding global volume datum f^*v as defined in §2.1.3.

We have the global relative Hecke operators

$$\Gamma_{c,f^*v}^e : H_c^*(\text{Bun}_G^e) \rightarrow H_c^{*+d_v-2}(\text{Bun}_G^{e+e_{\mathcal{K}}}) \quad (2.22)$$

and

$$\Gamma_{f^*v}^e : H^*(\text{Bun}_G^{e+e_{\mathcal{K}}}) \rightarrow H^{*+d_v-2}(\text{Bun}_G^e) \quad (2.23)$$

which are given by restricting those in §2.2.1 to one connected component.

Fix $I = \{1, 2, \dots, r\}$ for some $r \in \mathbb{Z}_{\geq 0}$ and choose a sequence of local volume data $v_I := (v_i = (\mathcal{K}_i \in \text{Shv}(\text{Hk}_G^{1,e_i}), \mathfrak{c}_i^1, \mathfrak{d}_i^1))_{i \in I}$ such that $d_{v_i} = d_{\mathfrak{c}_i} + d_{\mathfrak{d}_i} = 2$ for any $i \in I$. Let $e_{v_I} = \sum_i e_i \in \pi_1(G)$. Define the global relative Hecke operator attached to v_I

$$\Gamma_{c,f^*v_I}^e := \Gamma_{c,f^*v_1}^{e+e_{v_I}-e_1} \circ \dots \circ \Gamma_{c,f^*v_r}^e : H_c^*(\text{Bun}_G^e) \rightarrow H_c^*(\text{Bun}_G^{e+e_{v_I}}) \quad (2.24)$$

and its dual

$$\Gamma_{f^*v_I}^e := \Gamma_{f^*v_r}^e \circ \dots \circ \Gamma_{f^*v_1}^{e+e_{v_I}-e_1} : H^*(\text{Bun}_G^{e+e_{v_I}}) \rightarrow H^*(\text{Bun}_G^e). \quad (2.25)$$

Moreover, we make the following assumption:

Assumption 2.9. Each local volume datum v_i is defined over a subfield $E \subset k$ such that E/\mathbb{Q} is a finite extension. That is, the sheaf \mathcal{K}_i is defined over E and the maps $\mathfrak{c}_i^1, \mathfrak{d}_i^1$ are defined over E .

Define

$$\text{vol}(\text{Sht}_{G,I}^e, f^*v_I) := \begin{cases} 0 & , e_{v_I} \neq 0 \\ \text{tr}(\text{Frob} \circ \Gamma_{c,f^*v_I}^e, H_c^*(\text{Bun}_G^e)) & , e_{v_I} = 0 \end{cases}. \quad (2.26)$$

One can also define the volume for all connected components

$$\text{vol}(\text{Sht}_{G,I}, f^*v_I) := \text{tr}(\text{Frob} \circ \Gamma_{c,v_I}, H_c^*(\text{Bun}_G)). \quad (2.27)$$

Then one has $\text{vol}(\text{Sht}_{G,I}, f^*v_I) = \sum_{e \in \pi_1(G)} \text{vol}(\text{Sht}_{G,I}^e, f^*v_I)$.

Remark 2.10. The trace above should be understood as the summation $\text{tr}(\text{Frob} \circ \Gamma_{c,f^*v_I}, H_c^*(\text{Bun}_G)) = \sum_{i \in \mathbb{Z}} \text{tr}(\text{Frob} \circ \Gamma_{c,f^*v_I}, H_c^i(\text{Bun}_G))$, in which each vector space $H_c^i(\text{Bun}_G)$ is finite-dimensional. Under Assumption 2.9, this sum is absolutely convergent under any identification $\overline{\mathbb{Q}}_{\ell} \cong \mathbb{C}$. This can be proved in the same way as [FYZ26, Proposition 5.5.3].

Example 2.11. When taking $v_I = (v_{\lambda_1, \det}, \dots, v_{\lambda_r, \det})$ where each term is as in Example 2.2 for $\lambda_I \in X_*(T)^r$. We have $\text{vol}(\text{Sht}_{G,I}, f^*v_I) = \text{vol}(\text{Sht}_{G, \leq \lambda_I}, f_{\text{Sht}, I}^* \mathcal{L}_{\det, I})$ where the later is considered in Theorem 1.5.

2.4.2. *Ordering the eigenweights.* Recall the (multi-)set of eigenweights $\{\epsilon_{v,i} \mid 1 \leq i \leq n\}$ defined in (2.21). When G is almost simple, we refine this multiset by partially introducing an ordering using the decomposition $\mathbb{V} = \bigoplus_{i=1}^n \overline{\mathbb{Q}}_{\ell}(-d_i)$, where $\{d_i \mid 1 \leq i \leq n\}$ are as in Theorem 1.2.

When G is not of type D_n for n even, the numbers $\{d_i, 1 \leq i \leq n\}$ are distinct and we order them such that $d_1 < d_2 < \dots < d_n$. Define $\mathbb{V}_i := \overline{\mathbb{Q}}_{\ell}(-d_i)$, and we have $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_i$. We define $E_v|_{\mathbb{V}_i} = -\epsilon_{v,i} \cdot \text{id}_{\mathbb{V}_i}$. In this case, the matrix E_v is diagonal under the decomposition $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_i$.

When G is of type D_n for n even, we take $d_i = 2i$ for $1 \leq i \leq n-1$ and $d_n = n$. When $i \neq n/2, n$, we define $\mathbb{V}_i := \overline{\mathbb{Q}}_{\ell}(-d_i)$. When $i \in \{n/2, n\}$, the degree n part of \mathbb{V} is a two dimensional vector space. We do the following instead: The outer automorphism group $S_2 \cong \text{Out}(G)$ acts on \mathbb{V} . We define $\mathbb{V}_{n/2}$ as the S_2 -invariant part of the degree n part of \mathbb{V} , and define \mathbb{V}_n as the part that S_2 acts via the non-trivial character.⁶ We still have $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_i$. For $i \neq n/2, n$, we define $E_v|_{\mathbb{V}_i} = -\epsilon_{v,i} \cdot \text{id}_{\mathbb{V}_i}$. When $i \in \{n/2, n\}$, we only define the (multi-)set $\{\epsilon_{v,n/2}, \epsilon_{v,n}\}$ as the set of eigenvalues of the 2×2 -matrix $-E_v|_{\mathbb{V}_{n/2} \oplus \mathbb{V}_n}$. In this case, the matrix E_v is block diagonal under the decomposition $\mathbb{V} = \bigoplus_{i=1}^n \mathbb{V}_i$ with a unique 2×2 -block $E_v|_{\mathbb{V}_{n/2} \oplus \mathbb{V}_n}$.

⁵One has connected components decomposition $\text{Hk}_G^1 = \coprod_{e \in \pi_1(G)} \text{Hk}_G^{1,e}$

⁶The one-dimensional vector space \mathbb{V}_n is spanned by the *Pfaffian*.

2.4.3. *Main result.* Now we formulate a general version of Theorem 1.5. Take v_I as before. We make the following assumption.

Assumption 2.12. We assume that $\{E_{v_i} \in \text{End}(\mathbb{V})\}_{i \in I}$ pairwise commute. Here E_{v_i} are defined in (2.20).

Note that Assumption 2.12 is an empty assumption when G is not of type D_n for n even. When G is of type D_n for n even, Assumption 2.12 amounts to requiring the 2×2 -matrices $E_{v_i}|_{\mathbb{V}_{n/2} \oplus \mathbb{V}_n}$ to be mutually commutative. Under this assumption, we can simultaneously upper-triangularize the matrices $E_{v_i}|_{\mathbb{V}_{n/2} \oplus \mathbb{V}_n}$. We can order each set $\{\epsilon_{v_i, n/2}, \epsilon_{v_i, n}\}$ such that each number $\epsilon_{v_i, n}$ is an eigenvalue for a joint eigenvector of all $E_{v_i}|_{\mathbb{V}_{n/2} \oplus \mathbb{V}_n}$.

Define the differential operator on \mathbb{R}^n

$$D_{v_i} = (2g - 2)b_{v_i} - (\log q)^{-1} \sum_{j=1}^n \epsilon_{v_i, j} \partial_{s_j} \quad (2.28)$$

where b_{v_i} are defined in (2.17).

Theorem 2.13. Let $v_I = \{v_i\}_{i=1, \dots, n}$ be a collection of local volume data such that $d_{v_i} = 2$ for each $i \in I$ and $e_{v_i} = 0$. Under Assumption 2.12 and Assumption 2.9, for each $e \in \pi_1(G)$, we have

$$\text{vol}(\text{Sht}_{G, I}^e, f^* v_I) = q^{\dim \text{Bun}_G} \left(\left(\prod_{i=1}^r D_{v_i} \right) \mathcal{L}_{C, G}(s_1, \dots, s_n) \right) \Big|_{s_1 = \dots = s_n = 0}. \quad (2.29)$$

Under the case Example 2.11, Theorem 2.13 becomes Theorem 1.5.

The proof of Theorem 2.13 will be given in §3.

Remark 2.14. Assumption 2.12 can fail. See [Fen26, Remark 1.3.9] for a counterexample. This also follows from our computation of $E_{v_{\lambda, \det}}$ in the type D case in §5.1.7.

3. CALCULATING GLOBAL RELATIVE HECKE OPERATORS

This section is devoted to the proof of Theorem 2.13, which will be given in §3.4.

3.1. **Cohomology of Bun_G .** We now recall the Atiyah–Bott formula for the cohomology ring $H^*(\text{Bun}_G)$ and the filtrations on it as considered in [FYZZ26].

3.1.1. *Augmentation filtration.* Recall the connected component decomposition $\text{Bun}_G = \coprod_{e \in \pi_1(G)} \text{Bun}_G^e$ where $\pi_1(G)$ is the algebraic fundamental group of G . For each $e \in \pi_1(G)$, the cohomology ring $H^*(\text{Bun}_G^e)$ has a natural augmentation ideal $H^{>0}(\text{Bun}_G^e) \subset H^*(\text{Bun}_G^e)$. We get a decreasing augmentation filtration $\{F_{\text{aug}}^{\bullet} H^*(\text{Bun}_G^e)\}_{\bullet \in \mathbb{Z}_{\geq 0}}$ on $H^*(\text{Bun}_G^e)$ such that

$$F_{\text{aug}}^i H^*(\text{Bun}_G^e) := (H^{>0}(\text{Bun}_G^e))^i \subset H^*(\text{Bun}_G^e).$$

3.1.2. *Atiyah–Bott formula.* Consider the map

$$f : \text{Bun}_G \times C \rightarrow \mathbb{B}(L^+G \rtimes \text{Aut}(D)).$$

It induces a map⁷

$$\text{taut} : H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \otimes H_{\bullet}(C) \rightarrow H^*(\text{Bun}_G)$$

defined as

$$\text{taut}(\gamma \otimes z) = \int_z f^* \gamma$$

where $\int_z : H^i(C) \rightarrow k$ is the natural integration map and $\text{pr}_1^{\text{Bun}_G} : \text{Bun}_G \times C \rightarrow \text{Bun}_G$ is the projection.

For each $e \in \pi_1(G)$, one can further restrict to a single connected component and get

$$\text{taut}^e : H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \otimes H_{\bullet}(C) \rightarrow H^*(\text{Bun}_G^e).$$

We abbreviate $\text{taut}(\gamma \otimes z) = \gamma^z$. We call classes of the form γ^z *tautological classes*.

Note that the map

$$\text{taut}^e |_{H^*(\mathbb{B}G) \otimes H_{\bullet}(C)} : H^*(\mathbb{B}G) \otimes H_{\bullet}(C) \rightarrow H^*(\text{Bun}_G)$$

⁷We write $H_{\bullet}(C)$ instead of $H_*(C)$ to match the homological grading here with the Ran grading in §3.1.3.

preserves augmentation filtrations on both sides. Here, we are considering the filtration $F_{\text{aug}}^i(H^*(\mathbb{B}G) \otimes H_{\bullet}(C)) := F_{\text{aug}}^i H^*(\mathbb{B}G) \otimes H_{\bullet}(C)$ on the left-hand-side. Therefore, we get an induced homomorphism

$$\text{Gr}_{\text{aug}}^{\blacktriangleright}(\text{taut}^e |_{H^*(\mathbb{B}G) \otimes H_{\bullet}(C)}) : \text{Gr}_{\text{aug}}^{\blacktriangleright}(H^*(\mathbb{B}G) \otimes H_{\bullet}(C)) \rightarrow \text{Gr}_{\text{aug}}^{\blacktriangleright} H^*(\text{Bun}_G^e).$$

The Atiyah–Bott formula can be formulated as follows:

Theorem 3.1. *The map $\text{Gr}_{\text{aug}}^{\blacktriangleright}(\text{taut}^e |_{H^*(\mathbb{B}G) \otimes H_{\bullet}(C)})|_{\mathbb{V} \otimes H_{\bullet}(C)} : \mathbb{V} \otimes H_{\bullet}(C) \rightarrow \text{Gr}_{\text{aug}}^{\blacktriangleright} H^*(\text{Bun}_G^e)$ induces a ring isomorphism*

$$\text{AB}^e : \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)) \xrightarrow{\sim} \text{Gr}_{\text{aug}}^{\blacktriangleright} H^*(\text{Bun}_G^e).$$

Theorem 3.1 is first proved in [HS10] in positive characteristic. See [FYZ26, Theorem 4.2.8] for this formulation and a generalization.

3.1.3. Ran filtration. Now we introduce the Ran grading considered in [FYZ26, §5.5.4]. For each $e \in \pi_1(G)$, define the *Ran filtration* $\{F_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^e)\}_{\bullet \in \mathbb{Z}_{\geq 0}}$ such that $F_i^{\text{Ran}} H^*(\text{Bun}_G^e) \subset H^*(\text{Bun}_G^e)$ is spanned by those elements of the form $\gamma_1^{z_1} \cdots \gamma_k^{z_k}$ such that $\sum_i |z_i| \leq i$. Here, we have $\gamma_s \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, $z_s \in H_{\bullet}(C)$, and $|z_s|$ is the homological degree of z_s , and k is any non-negative integer.

Note that the Ran filtration is multiplicative, therefore, the associated graded $\text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^e)$ carries a natural commutative ring structure and a natural augmentation filtration $\{F_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^e)\}_{\blacktriangleright \in \mathbb{Z}_{\geq 0}}$.

3.2. Commutator relation. We first introduce the key ingredient in the proof of Theorem 2.13: Theorem 3.2, which is a special case of the commutator relations conjectured in [LW25, §4.5].

We have a commutative diagram

$$\begin{array}{ccccc} H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) & \xrightarrow{H^*(c^1)} & H_{-(*)+d_c}^{BM}(\text{Hk}_G^1 / \mathbb{B}(L^+G \rtimes \text{Aut}(D)), \mathcal{K}) & \xrightarrow{H^*(\partial^1)} & H^{*+d_v}(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \\ \downarrow f^* & & \downarrow f_{\text{Hk}}^* & & \downarrow f^* \\ H^*(\text{Bun}_G \times C) & \xrightarrow{H^*(c)} & H_{-(*)+d_c}^{BM}(\text{Hk}_G / \text{Bun}_G \times C, f_{\text{Hk}}^* \mathcal{K}) & \xrightarrow{H^*(\partial)} & H^{*+d_v}(\text{Bun}_G \times C) \end{array} \quad (3.1)$$

The map $f^* : R^W[\hbar] = H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \rightarrow H^*(\text{Bun}_G \times C)$ admits the following explicit description: Consider the map $\text{ev} : \text{Bun}_G \times C \rightarrow \mathbb{B}G$ and the induced map $\text{ev}^* : R^W = H^*(\mathbb{B}G) \rightarrow H^*(\text{Bun}_G \times C)$. We have

$$f^* \alpha = \text{ev}^* \alpha \quad (3.2)$$

for $\alpha \in R^W \subset R^W[\hbar]$ and

$$f^* \hbar = 1 \otimes c_1(T_C) = 1 \otimes (2 - 2g)\xi \in H^*(\text{Bun}_G \times C). \quad (3.3)$$

Here $\xi \in H^2(C)$ is the fundamental class of C .

Consider the maps

$$\overleftarrow{f} : \text{Hk}_G \times C \xrightarrow{\overleftarrow{h} \times \text{id}} \text{Bun}_G \times C \times C \xrightarrow{\text{pr}_{1,3}} \text{Bun}_G \times C \xrightarrow{f} \mathbb{B}(L^+G \rtimes \text{Aut}(D))$$

and

$$\overrightarrow{f} : \text{Hk}_G \times C \xrightarrow{\overrightarrow{h} \times \text{id}} \text{Bun}_G \times C \times C \xrightarrow{\text{pr}_{1,3}} \text{Bun}_G \times C \xrightarrow{f} \mathbb{B}(L^+G \rtimes \text{Aut}(D)).$$

Consider the map $\Delta_{\text{Hk}} = (\text{id}, l) : \text{Hk}_G \rightarrow \text{Hk}_G \times C$ where $l : \text{Hk}_G \rightarrow C$ is the map remembering the leg. We have a Gysin homomorphism

$$\Delta_{\text{Hk},!} : H^*(\text{Hk}_G) \rightarrow H^{*+2}(\text{Hk}_G \times C)$$

and the map $f_{\text{Hk}}^* : H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D))) \rightarrow H^*(\text{Hk}_G)$.

Theorem 3.2. *For any $\alpha \in H^i(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, we have*

$$\overleftarrow{f}^* \alpha - \overrightarrow{f}^* \alpha = \Delta_{\text{Hk},!} f_{\text{Hk}}^* [\alpha] \in H^i(\text{Hk}_G \times C) \quad (3.4)$$

where $[\alpha]$ is defined in (2.15).

The proof of Theorem 3.2 will be given in §3.5.

Remark 3.3. Theorem 3.2 has an equivalent formulation, Theorem 3.8. The proof of these two equivalent theorems confirms [LW25, Conjecture 4.45] in the special case that $X = \text{point}$ and one of V, W is the trivial representation. Since [LW25, Assumption 4.46] is not satisfied in Theorem 3.8, the method of proof in [LW25] is not applicable here.

Remark 3.4. It is obvious that both sides of (3.4) have the same image under the map $\Delta_{\text{Hk}}^* : H^i(\text{Hk}_G \times C) \rightarrow H^i(\text{Hk}_G)$. Indeed, we have $\Delta_{\text{Hk}}^*(\overleftarrow{f}^* \alpha - \overrightarrow{f}^* \alpha) = f_{\text{Hk}}^*(\overleftarrow{h}_1^* \alpha - \overrightarrow{h}_1^* \alpha) = f_{\text{Hk}}^*(\overleftarrow{h} \cdot [\alpha]) = c_1(T_C) \cdot f_{\text{Hk}}^*[\alpha] = \Delta_{\text{Hk}}^* \Delta_{\text{Hk},!} f_{\text{Hk}}^*[\alpha]$. This verifies the compatibility between the sign on both sides of (3.4) and the normalization $\overleftarrow{h} = c_1(T_D) \in H^2(\mathbb{B} \text{Aut}(D))$.

For later use, we note the following immediate corollary of Theorem 3.2:

Corollary 3.5. *For any $\alpha \in H^i(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$ and $z \in H_*(C)$, we have*

$$\overleftarrow{h}^*(\text{pr}_1^{\text{Bun}_G})^* \alpha^z - \overrightarrow{h}^*(\text{pr}_1^{\text{Bun}_G})^* \alpha^z = f_{\text{Hk}}^*[\alpha] \cup l^* \text{PD}(z) \quad (3.5)$$

where $\text{pr}_1^{\text{Bun}_G} : \text{Bun}_G \times C \rightarrow C$ is the projection to the first factor and $\overleftarrow{h}, \overrightarrow{h}, f_{\text{Hk}}$ are the maps in (2.4). The map $\text{PD} : H_*(C) \xrightarrow{\sim} H^{2-*}(C)$ is the isomorphism induced from the Poincaré duality characterized by $\int_{\xi} (\zeta \cup \text{PD}(z)) = \int_z \zeta$ for every $\zeta \in H^*(C)$.

Proof. Consider the map $\text{pr}_1^{\text{Hk}_G} : \text{Hk}_G \times C \rightarrow \text{Hk}_G$ (resp. $\text{pr}_2^{\text{Hk}_G} : \text{Hk}_G \times C \rightarrow C$) given by projection to the first (resp. second) factor. Note that the Gysin map can be written as

$$\Delta_{\text{Hk},!} = \Delta_{\text{Hk},!} \Delta_{\text{Hk}}^* (\text{pr}_1^{\text{Hk}_G})^* = (\text{pr}_1^{\text{Hk}_G})^* (-) \cup \Delta_{\text{Hk},!} 1 = \text{pr}_1^*(-) \cup (1 \otimes \xi - \beta + \xi \otimes 1)$$

where $\beta = \sum \zeta_i \otimes \zeta^i$ for a basis (ζ_i) of $H^1(C)$ with dual basis (ζ^i) satisfying $\zeta_i \cup \zeta^j = \delta_{ij} \xi$. The equation (3.4) can be rewritten as

$$\overleftarrow{f}^* \alpha - \overrightarrow{f}^* \alpha = (\text{pr}_1^{\text{Hk}_G})^* f_{\text{Hk}}^*[\alpha] \cup (1 \otimes \xi - \beta + \xi \otimes 1). \quad (3.6)$$

The desired identity follows from applying $\int_z = (\text{pr}_1^{\text{Hk}_G})_! (- \cup (\text{pr}_2^{\text{Hk}_G})^* \text{PD}(z))$ to the identity (3.6). \square

3.3. Comparing local and global relative Hecke operators. Now we apply Theorem 3.2 to compute the global relative Hecke operators Γ_v . We fix a local volume datum $v = (\mathcal{K}, \mathfrak{c}^1, \mathfrak{d}^1)$ and study the map Γ_v^e evaluated on tautological classes.

Using (3.5), for each $\alpha \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$ and $z \in H_*(C)$, we have

$$\begin{aligned} \Gamma_v(\alpha^z) &= (\text{pr}_1^{\text{Bun}_G})_! H^*(\mathfrak{d}) H^*(\mathfrak{c}) (\text{pr}_1^{\text{Bun}_G})^* \alpha^z \\ &= (\text{pr}_1^{\text{Bun}_G})_! H^*(\mathfrak{d}) (\overleftarrow{h}^*(\text{pr}_1^{\text{Bun}_G})^* \alpha^z \cdot H^*(\mathfrak{c}) 1) \\ &= (\text{pr}_1^{\text{Bun}_G})_! H^*(\mathfrak{d}) ((\overrightarrow{h}^*(\text{pr}_1^{\text{Bun}_G})^* \alpha^z + f_{\text{Hk}}^*[\alpha] \cup l^* \text{PD}(z)) \cdot H^*(\mathfrak{c}) 1) \\ &= (\text{pr}_1^{\text{Bun}_G})_! ((\text{pr}_1^{\text{Bun}_G})^* \alpha^z \cdot H^*(\mathfrak{d}) H^*(\mathfrak{c}) 1) + (\text{pr}_1^{\text{Bun}_G})_! H^*(\mathfrak{d}) (f_{\text{Hk}}^*[\alpha] \cup l^* \text{PD}(z)) \cdot H^*(\mathfrak{c}) 1 \\ &= \alpha^z \cdot (\text{pr}_1^{\text{Bun}_G})_! (f^* H^*(\mathfrak{d}^1) H^*(\mathfrak{c}^1) 1) + (\text{pr}_1^{\text{Bun}_G})_! (f^* H^*(\mathfrak{d}^1) ([\alpha] \cdot H^*(\mathfrak{c}^1) 1) \cup l^* \text{PD}(z)) \\ &= \alpha^z \cdot \Gamma_v^1(1)^\xi + \widetilde{\nabla}_v^1(\alpha)^z \\ &= (2g - 2)b_v \cdot \alpha^z + \widetilde{\nabla}_v^1(\alpha)^z \end{aligned} \quad (3.7)$$

where the map Γ_v^1 is defined in (2.13) and the map $\widetilde{\nabla}_v^1$ is defined in (2.16). This is the global counterpart of (2.18).

Similarly, for $\alpha_i \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, $z_i \in H_*(C)$, $i \in I$, we have

$$\begin{aligned} \Gamma_v \left(\prod_{i \in I} \alpha_i^{z_i} \right) &= \sum_{J \subset I} \pm \left(\prod_{j \in J} \alpha_j^{z_j} \right) \text{pr}_{1,!} (f^* H^*(\mathfrak{d}^1) \left(\prod_{i \in I \setminus J} [\alpha_i] \cdot H^*(\mathfrak{c}^1) 1 \right) \cup \prod_{i \in I \setminus J} \text{PD}(z_i)) \\ &= \sum_{J \subset I} \pm \left(\prod_{j \in J} \alpha_j^{z_j} \right) \Gamma_{(\prod_{i \in I \setminus J} [\alpha_i]) \cdot v}^1(1)^{\cap_{i \in I \setminus J} z_i}. \end{aligned} \quad (3.8)$$

Here, we define $z_i \cap z_j = \text{PD}^{-1}(\text{PD}(z_i) \cup \text{PD}(z_j))$.

This formula is parallel to [FYZ26, (5.5.13)] and is a global analogue of (2.19).

3.4. Global relative Hecke operators on associated graded. We have the following global analogue of Lemma 2.6:

Lemma 3.6. *The map $\Gamma_v^e : H^*(\text{Bun}_G^{e+e\kappa}) \rightarrow H^*(\text{Bun}_G^e)$ preserves the Ran filtration F_{\bullet}^{Ran} defined on both sides in §3.1.3. Moreover, the induced map on associated graded⁸ $\text{Gr}_{\bullet}^{\text{Ran}} \Gamma_v^e : \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^{e+e\kappa}) \rightarrow \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^e)$ is a derivation, which preserves the augmentation filtration $F_{\text{aug}}^{\blacktriangleright}$ on both sides.*

Proof. The first claim follows from $|z_i \cap z_j| \leq \min\{|z_i|, |z_j|\}$ for any $z_i, z_j \in H_{\bullet}(C)$. The second claim follows from the observation that only the terms in (3.8) satisfying $|J| \geq |I| - 1$ survive under $\text{Gr}_{\bullet}^{\text{Ran}} \Gamma_v^e$. The last claim, concerning the preservation of the augmentation filtration, follows from (3.7) and the Atiyah–Bott formula (Theorem 3.1). \square

By Lemma 3.6, we get a derivation

$$\text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} \Gamma_v^e : \text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^{e+e\kappa}) \rightarrow \text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^e).$$

By Theorem 3.1, we have $\text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} H^*(\text{Bun}_G^{e+e\kappa}) \cong \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C))$. Under this isomorphism, we can identify the map above as

$$\text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} \Gamma_v^e : \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)) \rightarrow \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)). \quad (3.9)$$

On the other hand, the local relative Hecke operator on the Gross motive (2.20) induces a derivation

$$\nabla_{E_v \otimes \text{id}} : \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)) \rightarrow \text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)),$$

defined by requiring that its restriction to generators is given by $\nabla_{E_v \otimes \text{id}}|_{\mathbb{V} \otimes H_{\bullet}(C)} = E_v \otimes \text{id}$, and extended to the symmetric algebra via the Leibniz rule. We have the following key observation, which is a global counterpart of Proposition 2.7, generalizing [FYZ26, Proposition 5.6.17]:

Proposition 3.7. *We have $\text{Gr}_{\text{aug}}^{\blacktriangleright} \text{Gr}_{\bullet}^{\text{Ran}} \Gamma_v^e = (2g - 2)b_v \cdot \text{id} + \nabla_{E_v \otimes \text{id}} \in \text{End}(\text{Sym}^{\blacktriangleright}(\mathbb{V} \otimes H_{\bullet}(C)))$.*

Proof. This is immediate from (3.8) and (3.7). \square

Proof of Theorem 2.13. The proof given in [FYZ26, §5.6.18] translates verbatimly after replacing [FYZ26, Proposition 5.6.17] by Proposition 3.7. \square

3.5. Proof of Theorem 3.2. In this section, we prove the key identity (3.4).

3.5.1. Multiplicativity in α . We first prove that (3.4) holds for $\alpha_1 \alpha_2$ if it holds for α_1 and α_2 . Indeed, assume (3.4) holds for α_1 and α_2 . We have

$$\begin{aligned} \overleftarrow{f}^*(\alpha_1 \alpha_2) - \overrightarrow{f}^*(\alpha_1 \alpha_2) &= \overleftarrow{f}^* \alpha_1 (\overleftarrow{f}^* \alpha_2 - \overrightarrow{f}^* \alpha_2) + (\overleftarrow{f}^* \alpha_1 - \overrightarrow{f}^* \alpha_1) \overrightarrow{f}^* \alpha_2 \\ &= \overleftarrow{f}^* \alpha_1 \Delta_{\text{Hk}, !} f_{\text{Hk}}^*[\alpha_2] + \Delta_{\text{Hk}, !} f_{\text{Hk}}^*[\alpha_1] \overrightarrow{f}^* \alpha_2 \\ &= \Delta_{\text{Hk}, !} (\overleftarrow{h}^* f^* \alpha_1 f_{\text{Hk}}^*[\alpha_2] + f_{\text{Hk}}^*[\alpha_1] \overrightarrow{h}^* f^* \alpha_2) \\ &= \Delta_{\text{Hk}, !} f_{\text{Hk}}^* (\overleftarrow{h}_1^* \alpha_1[\alpha_2] + [\alpha_1] \overrightarrow{h}_1^* \alpha_2) \\ &= \Delta_{\text{Hk}, !} f_{\text{Hk}}^*[\alpha_1 \alpha_2] \end{aligned}$$

This verifies (3.4) for $\alpha_1 \alpha_2$.

3.5.2. Functoriality for $G_1 \rightarrow G_2$. Assume we have a group homomorphism between split reductive groups $G_1 \rightarrow G_2$, which induces maps $\phi_1 : \mathbb{B}(L^+ G_1 \rtimes \text{Aut}(D)) \rightarrow \mathbb{B}(L^+ G_2 \rtimes \text{Aut}(D))$, $\phi : \text{Bun}_{G_1} \rightarrow \text{Bun}_{G_2}$, $\phi_{1, \text{Hk}} : \text{Hk}_{G_1}^1 \rightarrow \text{Hk}_{G_2}^1$, and $\phi_{\text{Hk}} : \text{Hk}_{G_1} \rightarrow \text{Hk}_{G_2}$. We claim that (3.4) for $\alpha \in H^*(\mathbb{B}(L^+ G_2 \times \text{Aut}(D)))$ implies (3.4) for $\phi_1^* \alpha \in H^*(\mathbb{B}(L^+ G_1 \times \text{Aut}(D)))$. Indeed, assume (3.4) holds for $\alpha \in H^*(\mathbb{B}(L^+ G_2 \times \text{Aut}(D)))$. We have

$$\begin{aligned} \overleftarrow{f}^*(\phi_1^* \alpha) - \overrightarrow{f}^*(\phi_1^* \alpha) &= (\phi_{\text{Hk}} \times \text{id}_C)^* (\overleftarrow{f}^* \alpha - \overrightarrow{f}^* \alpha) \\ &= (\phi_{\text{Hk}} \times \text{id}_C)^* \Delta_{\text{Hk}, !} f_{\text{Hk}}^*[\alpha] \\ &= \Delta_{\text{Hk}, !} f_{\text{Hk}}^* \phi_{1, \text{Hk}}^*[\alpha] \\ &= \Delta_{\text{Hk}, !} f_{\text{Hk}}^*[\phi_1^* \alpha] \end{aligned}$$

⁸The local analogue of taking associated graded with respect to the Ran filtration should be regarded as taking $-\otimes_{k[[h]]} k$.

This verifies (3.4) for $\phi_1^* \alpha$.

3.5.3. *Equivalent formulation.* Note that Theorem 3.2 is equivalent to the following:

Theorem 3.8. *For each $\mathcal{K} \in \text{Shv}(\text{Hk}_G^1)^\heartsuit$, $m \in H^*(\mathcal{K})$, and $\alpha \in H^*(\mathbb{B}(L^+G \rtimes \text{Aut}(D)))$, we have*

$$\overleftarrow{f}^* \alpha \text{pr}_1^* f_{\text{Hk}}^* m - \overrightarrow{f}^* \alpha \text{pr}_1^* f_{\text{Hk}}^* m = \Delta_{\text{Hk},!} f_{\text{Hk}}^*([\alpha]m) \in H^*(\text{Hk}_G \times C, \text{pr}_1^* f_{\text{Hk}}^* \mathcal{K}) \quad (3.10)$$

where $\text{pr}_1 : \text{Hk}_G \times C \rightarrow \text{Hk}_G$ is the projection to the first factor and $f_{\text{Hk}}^* : H^*(\text{Hk}_G^1, \mathcal{K}) \rightarrow H^*(\text{Hk}_G, f_{\text{Hk}}^* \mathcal{K})$ is the pullback map.

To see the equivalence, in one direction, since $\Delta_{\text{Hk},!} f_{\text{Hk}}^*([\alpha]m) = \Delta_{\text{Hk},!} f_{\text{Hk}}^*([\alpha]) \cdot \text{pr}_1^* f_{\text{Hk}}^* m$, we know that (3.10) can be obtained from the identity (3.4) by acting on $\text{pr}_1^* f_{\text{Hk}}^* m$. Therefore, Theorem 3.2 implies Theorem 3.8.

Conversely, assume Theorem 3.8 is true. To see (3.4), we only need to show that any element $\alpha \in H^*(\text{Hk}_G \times C)$ annihilating $\text{pr}_1^* f_{\text{Hk}}^* m \in IH^*(\text{Hk}_{G, \leq \lambda} \times C)$ for all $\lambda \in X_*(T)_+$ and $m \in IH^*(\text{Hk}_{G, \leq \lambda}^1)$ is zero. Since

$$H^*(\text{Hk}_G \times C) = \lim_{\lambda} H^*(\text{Hk}_{G, \leq \lambda} \times C),$$

we only need to show that for any $\lambda \in X_*(T)_+$, any element $\alpha \in H^*(\text{Hk}_{G, \leq \lambda} \times C)$ annihilating

$$\text{pr}_1^* f_{\text{Hk}}^* IH^*(\text{Hk}_{G, \leq \lambda}^1) \subset IH^*(\text{Hk}_{G, \leq \lambda} \times C)$$

is zero. By Künneth formula, we are reduced to show that the annihilator of

$$f_{\text{Hk}}^* IH^*(\text{Hk}_{G, \leq \lambda}^1) \subset IH^*(\text{Hk}_{G, \leq \lambda})$$

in $H^*(\text{Hk}_{G, \leq \lambda})$ is zero. Note that we have

$$H^*(\text{Hk}_{G, \leq \lambda}) = H^*(\text{Bun}_G \times C) \otimes_{R^W[\hbar]} H^*(\text{Hk}_{G, \leq \lambda}^1)$$

and

$$IH^*(\text{Hk}_{G, \leq \lambda}) = H^*(\text{Bun}_G \times C) \otimes_{R^W[\hbar]} IH^*(\text{Hk}_{G, \leq \lambda}^1).$$

Since the map $H^*(\text{Hk}_{G, \leq \lambda}^1) \rightarrow \text{End}_{R^W[\hbar]}(IH^*(\text{Hk}_{G, \leq \lambda}^1))$ is injective. We only need to show that

$$Q := \text{coker}(H^*(\text{Hk}_{G, \leq \lambda}^1) \rightarrow \text{End}_{R^W[\hbar]}(IH^*(\text{Hk}_{G, \leq \lambda}^1)))$$

is a free $R^W[\hbar]$ -module. By [Gin98, Theorem 1.5], we know

$$[H^*(\text{Hk}_{G, \leq \lambda}^1) \rightarrow \text{End}_{R^W[\hbar]}(IH^*(\text{Hk}_{G, \leq \lambda}^1))] \otimes_{R^W[\hbar]} k = [H^*(\text{Gr}_{G, \leq \lambda}) \rightarrow \text{End}(IH^*(\text{Gr}_{G, \leq \lambda}))]$$

is an injection. Since both $\text{End}_{R^W[\hbar]}(IH^*(\text{Hk}_{G, \leq \lambda}^1))$ and $H^*(\text{Hk}_{G, \leq \lambda}^1)$ are free $R^W[\hbar]$ -modules, we know that $\text{Tor}_{R^W[\hbar]}^{\leq -1}(Q, k) = 0$. Since Q is a finitely generated graded $R^W[\hbar]$ -module, this implies that Q is a free $R^W[\hbar]$ -module.

3.5.4. *Propagation through convolution.* We claim that Theorem 3.8 is stable under the convolution of $\text{Shv}(\text{Hk}_G^1)^\heartsuit$, that is, assuming (3.10) holds for $m_1 \in H^*(\mathcal{K}_1)$ and $m_2 \in H^*(\mathcal{K}_2)$ for $\mathcal{K}_1, \mathcal{K}_2 \in \text{Shv}(\text{Hk}_G^1)^\heartsuit$, one can deduce that (3.10) holds for $m_1 \otimes m_2 \in H^*(\mathcal{K}_1 * \mathcal{K}_2) \cong H^*(\mathcal{K}_1) \otimes_{R^W[\hbar]} H^*(\mathcal{K}_2)$.

Consider the global iterated Hecke stack $\widetilde{\text{Hk}}_G$ defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\text{Hk}}_G & \xrightarrow{\overrightarrow{p}} & \text{Hk}_G \\ \downarrow \overleftarrow{p} & & \downarrow \overleftarrow{h} \\ \text{Hk}_G & \xrightarrow{\overrightarrow{h}} & \text{Bun}_G \times C \end{array} \quad .$$

We have three maps $\overleftarrow{h}_2, h_2, \overrightarrow{h}_2 : \widetilde{\text{Hk}}_G \rightarrow \text{Bun}_G \times C$ defined by $\overrightarrow{h}_2 := \overrightarrow{h} \circ \overrightarrow{p}$, $\overleftarrow{h}_2 := \overleftarrow{h} \circ \overleftarrow{p}$, and $h_2 := \overleftarrow{h} \circ \overrightarrow{p} = \overrightarrow{h} \circ \overleftarrow{p}$. We also have a map $m : \widetilde{\text{Hk}}_G \rightarrow \text{Hk}_G$ given by composition of modifications. We have a map $\Delta_{\widetilde{\text{Hk}}} = (\text{id}, l \circ \overrightarrow{p}) : \widetilde{\text{Hk}} \rightarrow \widetilde{\text{Hk}} \times C$.

Similarly, we have the local iterated Hecke stack defined by the Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Hk}}_G^1 & \xrightarrow{\vec{p}_1} & \mathrm{Hk}_G^1 \\ \downarrow \overleftarrow{p}_1 & & \downarrow \overleftarrow{h}_1 \\ \mathrm{Hk}_G^1 & \xrightarrow{\vec{h}_1} & \mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)) \end{array}$$

and maps $\overleftarrow{h}_{1,2}, h_{1,2}, \vec{h}_{1,2} : \widetilde{\mathrm{Hk}}_G^1 \rightarrow \mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))$, $m_1 : \widetilde{\mathrm{Hk}}_G^1 \rightarrow \mathrm{Hk}_G^1$. We have a map $f_{\mathrm{Hk},2} : \widetilde{\mathrm{Hk}}_G \rightarrow \widetilde{\mathrm{Hk}}_G^1$. We have $\mathcal{K}_1 * \mathcal{K}_2 = m_{1,!}(\overleftarrow{p}_1^* \mathcal{K}_1 \otimes \overrightarrow{p}_1^* \mathcal{K}_2)$.

We would like to prove

$$\overleftarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2) - \overrightarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2) = \Delta_{\mathrm{Hk},!} f_{\mathrm{Hk}}^*([\alpha]m_1 \otimes m_2) \in H^*(\mathrm{Hk}_G \times C, \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(\mathcal{K}_1 * \mathcal{K}_2)).$$

Since we have an isomorphism $r_m : H^*(\mathrm{Hk}_G \times C, \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(\mathcal{K}_1 * \mathcal{K}_2)) \cong H^*(\widetilde{\mathrm{Hk}}_G \times C, \widetilde{\mathrm{pr}}_1^* f_{\mathrm{Hk},2}^*(\overleftarrow{p}_1^* \mathcal{K}_1 \otimes \overrightarrow{p}_1^* \mathcal{K}_2))$ where $\widetilde{\mathrm{pr}}_1 : \widetilde{\mathrm{Hk}}_G \times C \rightarrow \widetilde{\mathrm{Hk}}_G$ is the projection map to the first factor. The identity above is equivalent to

$$r_m(\overleftarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2) - \overrightarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2)) = r_m \Delta_{\mathrm{Hk},!} f_{\mathrm{Hk}}^*([\alpha]m_1 \otimes m_2).$$

Note that

$$\begin{aligned} & r_m(\overleftarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2) - \overrightarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^*(m_1 \otimes m_2)) \\ &= (\overleftarrow{p} \times \mathrm{id})^*(\overleftarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_1 - \overrightarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_1) \cup (\overrightarrow{p} \times \mathrm{id})^* \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_2 \\ &\quad + (\overleftarrow{p} \times \mathrm{id})^* \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_1 \cup (\overrightarrow{p} \times \mathrm{id})^*(\overleftarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_2 - \overrightarrow{f}^* \alpha \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_2) \\ &= (\overleftarrow{p} \times \mathrm{id})^* \Delta_{\mathrm{Hk},!} f_{\mathrm{Hk}}^*([\alpha]m_1) \cup (\overrightarrow{p} \times \mathrm{id})^* \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_2 + (\overleftarrow{p} \times \mathrm{id})^* \mathrm{pr}_1^* f_{\mathrm{Hk}}^* m_1 \cup (\overrightarrow{p} \times \mathrm{id})^* \Delta_{\mathrm{Hk},!} f_{\mathrm{Hk}}^*([\alpha]m_2) \\ &= \Delta_{\widetilde{\mathrm{Hk}},!}(\overleftarrow{p}^* f_{\mathrm{Hk}}^*([\alpha]m_1) \cup \overrightarrow{p}^* f_{\mathrm{Hk}}^* m_2 + \overleftarrow{p}^* f_{\mathrm{Hk}}^* m_1 \cup \overrightarrow{p}^* f_{\mathrm{Hk}}^*([\alpha]m_2)) \\ &= \Delta_{\widetilde{\mathrm{Hk}},!} f_{\mathrm{Hk},2}^*(\overleftarrow{p}_1^*([\alpha]m_1) \cup \overrightarrow{p}_1^* m_2 + \overleftarrow{p}_1^* m_1 \cup \overrightarrow{p}_1^*([\alpha]m_2)) \\ &= \Delta_{\widetilde{\mathrm{Hk}},!} f_{\mathrm{Hk},2}^* r_{m_1}([\alpha]m_1 \otimes m_2) \\ &= r_m \Delta_{\mathrm{Hk},!} f_{\mathrm{Hk}}^*([\alpha]m_1 \otimes m_2). \end{aligned}$$

Here, we are using the isomorphism $r_{m_1} : H^*(\mathrm{Hk}_G^1, \mathcal{K}_1 * \mathcal{K}_2) \cong H^*(\widetilde{\mathrm{Hk}}_G^1, \overleftarrow{p}_1^* \mathcal{K}_1 \otimes \overrightarrow{p}_1^* \mathcal{K}_2)$ and the identity $m_1^*[\alpha] = \overleftarrow{p}_1^*[\alpha] + \overrightarrow{p}_1^*[\alpha]$ in the next to the last step. This finishes the proof.

3.5.5. Conclude the proof. First, note that the images of $\phi_1^* : H^*(\mathbb{B}(L^+ \mathrm{GL}_N \times \mathrm{Aut}(D))) \rightarrow H^*(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)))$ for all homomorphisms $G \rightarrow \mathrm{GL}_N$ generate $H^*(\mathbb{B}(L^+G \rtimes \mathrm{Aut}(D)))$ as an algebra. By the functoriality and multiplicativity, we are reduced to proving Theorem 3.2 for GL_N . By the equivalent formulation using perverse sheaves and propagation under convolution, we are reduced to showing Theorem 3.8 when $\mathcal{K} = \mathrm{IC}_{(1,0,\dots,0)} \in \mathrm{Shv}(\mathrm{Hk}_{\mathrm{GL}_N}^1)^\heartsuit$. This case follows from [FYZ26, Theorem 3.2.1], which can also be verified via a straightforward computation.

4. LANGLANDS DUAL DESCRIPTION OF EIGENWEIGHTS

In this section, we explain the local volume datum associated with the determinant line bundle $\mathcal{L}_{\mathrm{det}} \in \mathrm{Pic}(\mathrm{Hk}_G^1)_{\mathbb{Q}}$ and a coweight $\lambda \in X_*(T)_+$ as promised in Example 2.2. Moreover, when G is simple, we give a description of the constants $b_\lambda, \epsilon_{\lambda,i}$ involved in Theorem 1.2 on the Langlands dual side.

4.1. Local volume datum for the determinant line bundle. We now consider a specific choice of local volume datum defined in Definition 2.1.

4.1.1. The determinant line bundle. We first recall the construction of the determinant line bundle $\mathcal{L}_{\mathrm{det}} \in \mathrm{Pic}(\mathrm{Hk}_G^1)_{\mathbb{Q}}$. It is characterized as follows. Let

$$q : L^+G \backslash LG \rightarrow \mathrm{Hk}_G^1$$

be the natural map. We require that $q^* \mathcal{L}_{\mathrm{det}} \in \mathrm{Pic}(L^+G \backslash LG) \subset \mathrm{Pic}(L^+G \backslash LG)_{\mathbb{Q}}$ is ample and generates $\mathrm{Pic}(L^+G \backslash LG^e)$ for each $e \in \pi_0(LG)$. Here, LG^e is the connected component of LG indexed by e . Moreover, $\mathcal{L}_{\mathrm{det}}$ is relatively ample with respect to the map

$$\vec{h}_1 : \mathrm{Hk}_G^1 \rightarrow \mathbb{B}(L^+G \rtimes \mathrm{Aut}(D))$$

in (2.4).

We now recall an explicit construction of \mathcal{L}_{\det} . For each finite-dimensional representation $V \in \text{Rep}(G)$ with representation map $r_V : G \rightarrow \text{GL}(V)$, we obtain an induced map

$$\phi_V : \text{Hk}_G^1 \rightarrow \text{Hk}_{\text{GL}(V)}^1.$$

We identify $\text{Hk}_{\text{GL}(V)}^1$ with the moduli stack of modifications $\{\mathcal{E}_1 \dashrightarrow \mathcal{E}_2\}$ of vector bundles of rank $\dim(V)$ over D considered modulo $\text{Aut}(D)$, and we adopt the convention that the map

$$\overrightarrow{h}_{\text{GL}(V),1} : \text{Hk}_{\text{GL}(V)}^1 \rightarrow \mathbb{B}(L^+ \text{GL}(V) \rtimes \text{Aut}(D))$$

sends $(\mathcal{E}_1 \dashrightarrow \mathcal{E}_2)$ to \mathcal{E}_2 . Let \mathcal{L}_{Std} be the line bundle on $\text{Hk}_{\text{GL}(V)}^1$ whose fiber at $(\mathcal{E}_1 \dashrightarrow \mathcal{E}_2)$ is

$$(\mathcal{E}_2 : \mathcal{E}_1) := \det(\mathcal{E}_2/\mathcal{E}_1 \cap \mathcal{E}_2) \otimes \det(\mathcal{E}_1/\mathcal{E}_1 \cap \mathcal{E}_2)^{-1}.$$

Define $\mathcal{L}_V := \phi_V^* \mathcal{L}_{\text{Std}} \in \text{Pic}(\text{Hk}_G^1)$.

Let $\text{Ad} \in \text{Rep}(G)$ denote the adjoint representation. The above construction yields a line bundle $\mathcal{L}_{\text{Ad}} \in \text{Pic}(\text{Hk}_G^1)$. We then define the determinant line bundle by

$$\mathcal{L}_{\det} := \mathcal{L}_{\text{Ad}}^{\otimes 1/(2h_G^\vee)} \in \text{Pic}(\text{Hk}_G^1)_{\mathbb{Q}},$$

where h_G^\vee is the dual Coxeter number of G .

4.1.2. *Local volume data.* For each dominant coweight $\lambda \in X_*(T)_+$, define

$$d_\lambda := \langle 2\rho, \lambda \rangle. \quad (4.1)$$

We use $\text{IC}_\lambda \in \text{Shv}(\text{Hk}_G^1)$ to denote the intersection complex of the closed Schubert cell $\text{Hk}_{G, \leq \lambda}^1 \subset \text{Hk}_G^1$ normalized such that $\text{IC}_\lambda \langle -d_\lambda \rangle|_{\text{Hk}_{G, \leq \lambda}^1}$ lies in the heart of the naive t -structure. Under the isomorphism

$$\text{Hom}^0(\overleftarrow{h}_{1,!} \text{IC}_\lambda \langle d_\lambda \rangle, \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}) \cong \text{Hom}^0(\overleftarrow{h}_{1,!} \underline{k}_{\text{Hk}_{G, \leq \lambda}^1} \langle 2d_\lambda \rangle, \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}) \cong k, \quad (4.2)$$

we use $[\text{Hk}_{G, \leq \lambda}^1]^{BM} \in \text{Hom}^0(\overleftarrow{h}_{1,!} \text{IC}_\lambda \langle d_\lambda \rangle, \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))})$ to denote the element corresponding to $1 \in k$, which is the (relative) fundamental (Borel-Moore homology) class of $\text{Hk}_{G, \leq \lambda}^1$ (over \overleftarrow{h}_1).

Consider the first Chern class $c_1(\mathcal{L}_{\det}) \in H^2(\text{Hk}_G^1)$. We define

$$\mathbf{c}_{\lambda, \det}^1 := c_1(\mathcal{L}_{\det})^{d_\lambda+1} \cdot [\text{Hk}_{G, \leq \lambda}^1]^{BM} \in \text{Corr}_{\text{Hk}_G^1, \text{IC}_\lambda \langle -d_\lambda - 2 \rangle}(\underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}, \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}).$$

In other words, the element

$$\mathbf{c}_{\lambda, \det}^1 \in \text{Hom}^0(\overleftarrow{h}_{1,!} \text{IC}_\lambda \langle -d_\lambda - 2 \rangle, \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))})$$

is given by the composition

$$\overleftarrow{h}_{1,!} \text{IC}_\lambda \langle -d_\lambda - 2 \rangle \xrightarrow{\overleftarrow{h}_{1,!} (-\cup_{c_1(\mathcal{L}_{\det})^{d_\lambda+1}})} \overleftarrow{h}_{1,!} \text{IC}_\lambda \langle d_\lambda \rangle \xrightarrow{[\text{Hk}_{G, \leq \lambda}^1]^{BM}} \underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}.$$

Consider also the (relative) fundamental (cohomology) class (over \overrightarrow{h}_1)

$$\mathbf{d}_\lambda^1 := [\text{Hk}_{G, \leq \lambda}^1] \in \text{Hom}^0(\underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}, \overrightarrow{h}_{1,!} \text{IC}_\lambda \langle -d_\lambda \rangle),$$

that is, the element corresponds to $1 \in k$ under the isomorphism

$$\text{Hom}^0(\underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}, \overrightarrow{h}_{1,!} \text{IC}_\lambda \langle -d_\lambda \rangle) \cong \text{Hom}^0(\underline{k}_{\mathbb{B}(L+G \rtimes \text{Aut}(D))}, \overrightarrow{h}_{1,*} \underline{k}_{\text{Hk}_{G, \leq \lambda}^1}) \cong k.$$

Definition 4.1. The local volume datum given by the determinant line bundle is the triple

$$v_{\lambda, \det} = (\text{IC}_\lambda, \mathbf{c}_{\lambda, \det}^1, \mathbf{d}_\lambda^1)$$

in which each term is defined as above.

4.2. **Translation to the spectral side.** In this section, we describe the constants b_λ and $\epsilon_{\lambda, i}$ appearing in Theorem 1.2 in terms of the Langlands dual group. For simplicity, we assume that G is almost simple.

4.2.1. *Notations.* Let $\kappa_{\min} : \check{\mathfrak{g}} \times \check{\mathfrak{g}} \rightarrow k$ be the \check{G} -invariant non-degenerate bilinear form normalized such that $\kappa_{\min}(\alpha_{\text{long}}, \alpha_{\text{long}}) = 2$ where $\alpha_{\text{long}} \in X_*(\check{T})$ is any long coroot of \check{G} .

For each finite-dimensional representation $V \in \text{Rep}(\check{G})$, let $r_V : \check{G} \rightarrow \text{GL}(V)$ be the representation map and $dr_V : \check{\mathfrak{g}} \rightarrow \mathfrak{gl}(V)$ its differential. This defines a \check{G} -invariant bilinear form

$$\kappa_V : \check{\mathfrak{g}} \times \check{\mathfrak{g}} \rightarrow k, \quad \kappa_V(X, Y) = \text{tr}(dr_V(X) dr_V(Y)) \quad (4.3)$$

for $X, Y \in \check{\mathfrak{g}}$. Let $\text{Ad} \in \text{Rep}(\check{G})$ denote the adjoint representation. Then

$$\kappa_{\min} = \frac{\kappa_{\text{Ad}}}{2h_G^{\vee} l_G}. \quad (4.4)$$

Here l_G is the lacing number, equal to 1 for types ADE , 2 for types BCF , and 3 for type G , and h_G^{\vee} is the dual Coxeter number of $\check{\mathfrak{g}}$.

We use $\check{\alpha}_j \in X^*(\check{T})$, $1 \leq j \leq n$ to denote the simple roots of \check{G} . By restriction, we can regard κ_{\min} as a W -invariant bilinear form on $\check{\mathfrak{t}}$ (hence also $\mathfrak{t} \cong \check{\mathfrak{t}}^*$) satisfying $\kappa_{\min}(\check{\alpha}_{\text{short}}, \check{\alpha}_{\text{short}}) = 2$ where $\check{\alpha}_{\text{short}} \in X^*(\check{T})$ is any short root of \check{G} . That is, the bilinear form κ_{\min} corresponds to the basic bilinear form on $\check{\mathfrak{g}}$.

We use $\check{\mathfrak{g}}_{\mathbb{Z}}$ to denote the Chevalley form of $\check{\mathfrak{g}}$. Choose Chevalley generators $x_i \in \check{\mathfrak{g}}_{\mathbb{Z}, \alpha_i}$, $1 \leq i \leq n$ which are unique up to signs. Let $e = \sum_{i=1}^n \kappa_{\min}(\check{\alpha}_i, \check{\alpha}_i) x_i / 2 \in \check{\mathfrak{g}}_{\mathbb{Z}}$. Let $\rho \in X_*(\check{T})_{\mathbb{Q}} \subset \check{\mathfrak{t}}$ be the half sum of all positive coroots of \check{G} . This determines an \mathfrak{sl}_2 -triple $(e, 2\rho, f)$ for a unique $f \in \check{\mathfrak{g}}$.

Using the non-degenerate bilinear form κ_{\min} , we make identification $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$. Consider the Kostant slice $\check{\mathfrak{s}} = e + \check{\mathfrak{g}}_f \subset \check{\mathfrak{g}}$. We have a canonical grading $\check{\mathfrak{g}}_f = \bigoplus_d \check{\mathfrak{g}}_{f,d}$ where $[\rho, -]$ restricts to $\cdot d$ on $\check{\mathfrak{g}}_{f,d}$.⁹ Then we have $\check{\mathfrak{g}}_f = \bigoplus_{j=1}^n \check{\mathfrak{g}}_{f,-d_j+1}$ when G is not of type D_n and n is even.

Let $\check{\mathfrak{c}} = \check{\mathfrak{g}} // \check{G}$ be the Chevalley quotient and $\chi : \check{\mathfrak{g}} \rightarrow \check{\mathfrak{c}}$ be the quotient map. It induces an isomorphism $\chi|_{\check{\mathfrak{s}}} : \check{\mathfrak{s}} \xrightarrow{\sim} \check{\mathfrak{c}}$. Moreover, we have $\check{\mathfrak{s}} \cong \check{\mathfrak{c}} \cong \mathbb{V}$ as graded vector spaces after properly normalizing the grading.

Let $J_{\check{G}} \rightarrow \check{\mathfrak{s}}$ be the regular centralizer whose fiber at $X \in \check{\mathfrak{s}}$ is the centralizer subgroup $C_{\check{G}}(X) \subset \check{G}$. The map χ induces a map $d\chi : \check{\mathfrak{g}} \times_{\check{\mathfrak{c}}} T^*\check{\mathfrak{c}} \rightarrow T^*\check{\mathfrak{g}} \cong \check{\mathfrak{g}} \times \check{\mathfrak{g}}^* \cong \check{\mathfrak{g}} \times \check{\mathfrak{g}}$ which restricts to an isomorphism $d\chi|_{\check{\mathfrak{s}}} : T^*\check{\mathfrak{s}} \xrightarrow{\sim} \text{Lie } J_{\check{G}}$.

The bilinear form κ_{\min} restricts to a non-degenerate bilinear form $\kappa := \kappa_{\min}|_{\check{\mathfrak{g}}_f \times \check{\mathfrak{g}}_e} : \check{\mathfrak{g}}_f \times \check{\mathfrak{g}}_e \rightarrow k$. Using this, we make identification $\check{\mathfrak{g}}_e \cong \check{\mathfrak{g}}_f^*$. Note that we also have a grading $\check{\mathfrak{g}}_e = \bigoplus_d \check{\mathfrak{g}}_{e,d}$ such that we have $(\check{\mathfrak{g}}_{e,-d})^* \cong \check{\mathfrak{g}}_{f,d}$.

We use $V_{\lambda} \in \text{Rep}(\check{G})$ to denote the irreducible representation of \check{G} with highest weight $\lambda \in X_*(T)_+$. We use $V_{\lambda, \mathbb{Z}} \subset V_{\lambda}$ to denote the standard representation of $\check{G}_{\mathbb{Z}}$ with highest weight λ , which is a \mathbb{Z} -lattice inside V_{λ} . We use $u_{\lambda,-} \in V_{\lambda, \mathbb{Z}}$ to denote a generator of the lowest weight submodule, and $u_{\lambda,+}^* \in V_{\lambda, \mathbb{Z}}^*$ to denote a generator of the lowest weight submodule of $V_{\lambda, \mathbb{Z}}^* = \text{Hom}_{\mathbb{Z}}(V_{\lambda, \mathbb{Z}}, \mathbb{Z})$. The generators $u_{\lambda,-}$ and $u_{\lambda,+}^*$ are normalized such that $\langle e^{d\lambda} \cdot u_{\lambda,-}, u_{\lambda,+}^* \rangle \geq 0$. This determines the pair $u_{\lambda,-}, u_{\lambda,+}^*$ up to a simultaneous sign change. Let $V_{\lambda} = \bigoplus_{\mu \in X^*(\check{T})} V_{\lambda}(\mu)$ be the weight space decomposition.

4.2.2. *Spectral description of $\epsilon_{\lambda,j}$.* Now we describe the eigenweights $\epsilon_{\lambda,j}$ involved in Theorem 1.2 (see also Example 2.8) as some invariants attached to the representation $V_{\lambda} \in \text{Rep}(\check{G})$.

Consider the bilinear form $\kappa_{\lambda} : \check{\mathfrak{g}}_f \times \check{\mathfrak{g}}_e \rightarrow k$ defined by

$$\kappa_{\lambda}(X, Y) = \left\langle \sum_{s=0}^{d_{\lambda}} e^s X Y e^{d_{\lambda}-s} \cdot u_{\lambda,-}, u_{\lambda,+}^* \right\rangle. \quad (4.5)$$

This determines an endomorphism $E_{\lambda} \in \text{End}(\check{\mathfrak{g}}_f) = \text{End}(\check{\mathfrak{s}})$ defined such that $\kappa(E_{\lambda}(Y), X) = \kappa_{\lambda}(Y, X)$ for any $X \in \check{\mathfrak{g}}_e$ and $Y \in \check{\mathfrak{g}}_f$.

Theorem 4.2. *Under the natural identification $\mathbb{V} \cong \check{\mathfrak{s}}$, we have $E_{v_{\lambda, \text{det}}} = -E_{\lambda}$.*

Proof. During this proof, we exchange the role of \overleftarrow{h} and \overrightarrow{h} everywhere to match the convention in most literature, like [BF08][YZ11]. In particular, we change the definition of \mathcal{L}_{det} such that it is relatively ample over \overleftarrow{h}_1 . Since $E_{\lambda} = E_{-w_0(\lambda)}$, we can still work with λ instead of $-w_0(\lambda)$.

⁹The d 's appearing here are the negative of exponents of $\check{\mathfrak{g}}$.

Recall that we use κ_{\min} to make identification $\check{c}^* \cong \check{c}$. By [BF08, Theorem 1(b)], there is a canonical graded isomorphism $H^*(\mathrm{Hk}_G^1) \cong \bigoplus_{z \in Z(\check{G})} \mathcal{O}(D_{\check{c}}(\check{c} \times \check{c}))$.¹⁰ Here, we use $D_{\check{c}}(\check{c} \times \check{c})$ to denote the deformation to the normal cone of the scheme $\check{c} \times \check{c}$ along the closed subscheme $\check{c} \subset \check{c} \times \check{c}$ embedding along the diagonal. Under this identification, the natural maps $\overleftarrow{h}_1^*, \overrightarrow{h}_1^* : H^*(\mathbb{B}(L^+G \times \mathrm{Aut}(D))) \rightarrow H^*(\mathrm{Hk}_G^1)$ get identified with pull-back maps for the natural projections $p_1, p_2 : D_{\check{c}}(\check{c} \times \check{c}) \rightarrow \check{c} \times \mathbb{A}^1$ where we identify $\mathbb{A}^1 \cong \mathrm{Spec} k[\hbar]$.

The canonical isomorphism $H^*(\mathrm{Hk}_G^1) \otimes_{k[\hbar]} k \cong H^*(L^+G \backslash LG / L^+G)$ gets identified with the isomorphism $\bigoplus_{z \in Z(\check{G})} \mathcal{O}(D_{\check{c}}(\check{c} \times \check{c}) \times_{\mathbb{A}^1} \{0\}) \cong \bigoplus_{z \in Z(\check{G})} \mathcal{O}(T\check{c}) \cong \bigoplus_{z \in Z(\check{G})} U(\mathrm{Lie} J_{\check{G}})$. Under this identification, for $x \in \mathcal{O}(\check{c})$, the image of $[x] = (\overleftarrow{h}_1^* x - \overrightarrow{h}_1^* x) / \hbar$ in $H^*(L^+G \backslash LG / L^+G)$ is the differential 1-form $-dx \in \Gamma(\check{c}, \Omega_{\check{c}}) \subset \mathcal{O}(T\check{c}) \subset \bigoplus_{z \in Z(\check{G})} \mathcal{O}(T\check{c})$ where the last inclusion is via the direct summand indexed by the identity element $e \in Z(\check{G})$.¹¹ The first Chern class $c_1(\mathcal{L}_{\mathrm{det}}) \in H^*(L^+G \backslash LG / L^+G)$ is identified with the tautological section $e^{\mathrm{univ}} := (X \in \check{\mathfrak{s}} \mapsto X \in \mathrm{Lie} J_{\check{G}}) \in \mathrm{Lie} J_{\check{G}} \subset U(\mathrm{Lie} J_{\check{G}})$.

By [BF08, Theorem 4], the graded $H^*(L^+G \backslash LG / L^+G)$ -module $IH^*(L^+G \backslash \mathrm{Gr}_{G, \leq \lambda})$ gets identified with the graded vector space $V_{\lambda} \otimes \mathcal{O}(\check{\mathfrak{s}})$ equipped with the tautological graded $\bigoplus_{z \in Z(\check{G})} U(\mathrm{Lie} J_{\check{G}})$ -action. Under this identification, we can choose the lowest weight generators properly such that $[\mathrm{Hk}_{G, \leq \lambda}^1] \bmod \hbar = u_{\lambda, -} \otimes 1 \in V_{\lambda} \otimes \mathcal{O}(\check{\mathfrak{s}})$ and $[\mathrm{Hk}_{G, \leq \lambda}^1]^{BM} \bmod \hbar = u_{\lambda, +}^* \otimes 1 \in V_{\lambda}^* \otimes \mathcal{O}(\check{\mathfrak{s}})$.

Therefore, the map $E_{v_{\lambda, \mathrm{det}}} : \check{\mathfrak{s}} \rightarrow \check{\mathfrak{s}}$ is determined as follows: For $x \in \check{\mathfrak{s}} \cong \check{\mathfrak{s}}^*$, we have

$$\langle (e^{\mathrm{univ}})^{d_{\lambda}+1} (-dx) \cdot (u_{\lambda, -} \otimes 1), u_{\lambda, +}^* \otimes 1 \rangle = E_{v_{\lambda, \mathrm{det}}}(x) + \text{other terms not in } \check{\mathfrak{s}}^* \subset \mathcal{O}(\check{\mathfrak{s}}). \quad (4.6)$$

To simplify notations, we assume G is not of type D_{2n} . (This case can be treated similarly) Under this assumption, we have $\check{\mathfrak{g}}_f = \bigoplus_{j=1}^n \check{\mathfrak{g}}_{f, -d_j+1}$ and each graded piece $\check{\mathfrak{g}}_{f, -d_j+1}$ is one-dimensional. Therefore, we can choose non-zero elements $f_j \in \check{\mathfrak{g}}_{f, -d_j+1}$, and we have $\check{\mathfrak{g}}_{f, -d_j+1} = \mathrm{Span}(f_j)$. We have $e^{\mathrm{univ}} = e + \sum_j a_j f_j$ where $a_j \in (\check{\mathfrak{g}}_{f, -d_j+1})^* \subset \check{\mathfrak{s}}^*$ are homogeneous generators of $\mathcal{O}(\check{\mathfrak{s}})$ and $f_j \in \check{\mathfrak{g}}_{f, -d_j+1}$. Since $e^{d_{\lambda}+1}$ acts by zero on $V_{\lambda} \otimes \mathcal{O}(\check{\mathfrak{s}})$, from (4.6), we get

$$E_{v_{\lambda, \mathrm{det}}}(a_j) = - \langle \sum_{s=0}^{d_{\lambda}+1} e^s f_j e^{d_{\lambda}+1-s} (da_j)|_{e \in \check{\mathfrak{s}}} \cdot u_{\lambda, -}, u_{\lambda, +}^* \rangle \cdot a_j \quad (4.7)$$

where $(da_j)|_{e \in \check{\mathfrak{s}}} \in \check{\mathfrak{g}}_{e, d_j-1}$. Therefore, we only need to show that $\kappa_{\min}((da_j)|_{e \in \check{\mathfrak{s}}}, f_j) = 1$. This translates to the fact that a_j regarded as a function on $\check{\mathfrak{g}}$ has derivative at $e \in \check{\mathfrak{g}}$ along direction f_j equals 1, which follows from the tautological relation $a_j(e + \alpha f_j) = \alpha$ for $\alpha \in k$. \square

Theorem 4.2 has the following immediate corollary:

Corollary 4.3. *When G is not of type D_n for even n , we have $\epsilon_{\lambda, j} = \frac{\kappa_{\lambda} |_{\check{\mathfrak{g}}_{f, -d_j+1} \otimes \check{\mathfrak{g}}_{e, d_j-1}}}{\kappa_{\min} |_{\check{\mathfrak{g}}_{f, -d_j+1} \otimes \check{\mathfrak{g}}_{e, d_j-1}}} \in \mathbb{Q}$ for $1 \leq j \leq n$.*

4.2.3. *Spectral description of b_{λ} .* In this subsection, we describe the number $b_{\lambda} \in \mathbb{Q}$ involved in Theorem 1.2 (see also Example 2.8) as an invariant attached to the representation $V_{\lambda} \in \mathrm{Rep}(\check{G})$.

Theorem 4.4. *We have*

$$b_{\lambda} = \left\langle \sum_{s=0}^{d_{\lambda}} e^s H_2 e^{d_{\lambda}-s} u_{\lambda, -}, u_{\lambda, +}^* \right\rangle \in \mathbb{Q}$$

where $H_2 = (\sum_{\alpha \in \Phi_G} (d\alpha)^2) / 4h_G^{\vee} \in U(\check{\mathfrak{t}}) \subset U(\check{\mathfrak{g}})$, h_G^{\vee} is the dual Coxeter number of G .

Proof. As we did in the proof of Theorem 4.2, we exchange the role of \overleftarrow{h} and \overrightarrow{h} everywhere.

¹⁰By $\pi_0(\mathrm{Hk}_G^1) \cong \pi_1(G) \cong X^*(Z(\check{G}))$, we get a connected component decomposition $\mathrm{Hk}_G^1 = \coprod_{\chi \in X^*(Z(\check{G}))} \mathrm{Hk}_G^{1, \chi}$. Note that different $H^*(\mathrm{Hk}_G^{1, \chi})$ for $\chi \in X^*(Z(\check{G}))$ are canonically isomorphic. This gives a canonical diagonal embedding $H^*(\mathrm{Hk}_G^{1, \mathrm{triv}}) \subset H^*(\mathrm{Hk}_G^1)$ for $\mathrm{triv} \in X^*(Z(\check{G}))$. Under the normalization we choose, this embedding corresponds to $\mathcal{O}(D_{\check{c}}(\check{c} \times \check{c})) \subset \bigoplus_{z \in Z(\check{G})} \mathcal{O}(D_{\check{c}}(\check{c} \times \check{c}))$ which is the inclusion via the direct summand indexed by the identity element $e \in Z(\check{G})$.

¹¹To see this subtle minus sign here, consider the filtration $F_{\geq \mu} IH^*(T \backslash \mathrm{Gr}_{G, \leq \lambda} / \mathbb{G}_m) \subset IH^*(T \backslash \mathrm{Gr}_{G, \leq \lambda} / \mathbb{G}_m)$ as in the proof of Theorem 4.4. One can check that taking cup product with the image of $[x]$ in $H^*(T \backslash \mathrm{Gr}_G / \mathbb{G}_m)$ coincides with the multiplication by $-\partial_{\lambda} x \in \mathcal{O}(\check{\mathfrak{t}})$ on the graded piece $\mathrm{Gr}_{\mu}^F IH^*(T \backslash \mathrm{Gr}_{G, \leq \lambda} / \mathbb{G}_m)$ where $\partial_{\lambda} x$ is the partial derivative of $x \in \mathcal{O}(\check{\mathfrak{t}}) \subset \mathcal{O}(\check{\mathfrak{t}})$ along $\lambda \in \check{\mathfrak{t}} \cong \check{\mathfrak{t}}$.

Note that we can work with $H^*(\mathrm{Gr}_G/\mathbb{G}_m)$ instead of $H^*(\mathrm{Hk}_G^1)$ where \mathbb{G}_m acts on Gr_G via loop rotation. By the proof of [YZ11, Lemma 2.2], there is a canonical isomorphism $IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m) \cong V_\lambda \otimes k[\hbar]$. Moreover, let $T_\mu = LN^{-t^\mu}L^+G/L^+G \subset \mathrm{Gr}_G$ and \overline{T}_μ be its closure for $\mu \in X_*(T)$. Define

$$F^{\geq\mu}IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m) = \mathrm{im}(IH^*_{(\overline{T}_\mu \cap \mathrm{Gr}_{G,\leq\lambda})/\mathbb{G}_m}(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m) \rightarrow IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m)).$$

This gives a filtration $\{F^{\geq\mu}IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m) \subset IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m)\}_{\mu \in X_*(T)}$. On the spectral side, this filtration corresponds to the obvious filtration by weights $\{(G^{\geq\mu}V_\lambda) \otimes k[\hbar]\}_{\mu \in X_*(\check{T})}$. Passing to associated graded pieces, this isomorphism gives $\mathrm{Gr}^\mu F(IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m)) \cong i_\mu^* j_\mu^! \mathrm{IC}_{\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m} \otimes H^*(\mathbb{B}\mathbb{G}_m) \cong V_\lambda(\mu) \otimes k[\hbar]$. Here $i_\mu : \{t_\mu\}/\mathbb{G}_m \rightarrow T_\mu/\mathbb{G}_m$ and $j_\mu : T_\mu/\mathbb{G}_m \rightarrow \mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m$ are the natural inclusions.

Note that taking cup product with $c_1(\mathcal{L}_{\mathrm{det}})$ preserves the filtration $F^{\geq\mu}IH^*((L^+G \setminus LG_{\leq\lambda})/\mathbb{G}_m)$. Therefore, we get an induced endomorphism $\cup_{c_1(\mathcal{L}_{\mathrm{det}})}$ on the associated graded pieces $\mathrm{Gr}^\mu F(IH^*(\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m))$, which coincides with $\cup_{c_1(\mathcal{L}_{\mathrm{det}}|_{\{t^\mu\}/\mathbb{G}_m})}$ on $i_\mu^* j_\mu^! \mathrm{IC}_{\mathrm{Gr}_{G,\leq\lambda}/\mathbb{G}_m} \otimes H^*(\mathbb{B}\mathbb{G}_m)$. Note that $[\mathcal{L}_{\mathrm{det}}|_{\{t^\mu\}/\mathbb{G}_m}] \in \mathrm{Pic}(\mathbb{B}\mathbb{G}_m)_\mathbb{Q} \cong \mathbb{Q}$ can be identified with $-(\sum_{\alpha \in \Phi_G} \langle \alpha, \mu \rangle^2) / 4h_G^\vee = -\kappa_{\min}(\mu, \mu) / 2$.

On the other hand, we know that $\cup_{c_1(\mathcal{L}_{\mathrm{det}})}$ modulo \hbar can be identified with the action of $e \in \check{\mathfrak{g}}$ on V_λ . Unwinding the construction, we know b_λ is the coefficient of \hbar in $-\langle c_1(\mathcal{L}_{\mathrm{det}})^{d_\lambda+1} \cdot u_{\lambda,-}, u_{\lambda,+}^* \rangle \in k[\hbar]$. The desired identity follows immediately. \square

Remark 4.5. Here is another description of E_λ and b_λ , which reveals their relation to relative Langlands duality as developed in [BZSV24]. We follow the notation of [LW25, §4]. Let $\check{M} = T_\psi^* \check{N} \cong \check{G} \times \check{\mathfrak{s}}$ be the \check{G} -Hamiltonian space corresponding to the Whittaker period. It is the relative Langlands dual of the trivial G -Hamiltonian space $M = T^*X$ for $X = *$. The elements $\mathfrak{c}_{\lambda,\mathrm{det}}^1$ and \mathfrak{d}_λ^1 give rise to maps $\alpha : V_\lambda \rightarrow \mathrm{PL}_{X,\hbar}$ and $\beta : V_\lambda^* \rightarrow \mathrm{PL}_{X,\hbar}$. Reducing modulo \hbar and using the isomorphism $\mathrm{PL}_{X,\hbar} \otimes_{k[\hbar]} k \cong \mathcal{O}(\check{M})$, we obtain maps $\bar{\alpha} : V_\lambda \rightarrow \mathcal{O}(\check{M})$ and $\bar{\beta} : V_\lambda^* \rightarrow \mathcal{O}(\check{M})$. Consider the canonical element unit $\in V_\lambda \otimes V_\lambda^*$. Then $m \circ (\alpha \otimes \beta)(\mathrm{unit}) = b_\lambda \cdot \hbar \in \mathrm{PL}_{X,\hbar}$, where $m : \mathrm{PL}_{X,\hbar} \otimes \mathrm{PL}_{X,\hbar} \rightarrow \mathrm{PL}_{X,\hbar}$ is the multiplication map. Define a vector field $X_\lambda := \bar{m} \circ (d \otimes \mathrm{id}) \circ (\bar{\alpha} \otimes \bar{\beta})(\mathrm{unit}) \in \Gamma(\check{M}, T^*\check{M}) \cong \Gamma(\check{M}, T\check{M})$, where $d : \mathcal{O}(\check{M}) \rightarrow \Gamma(\check{M}, T^*\check{M})$ is the de Rham differential and $\bar{m} : \mathcal{O}(\check{M}) \otimes \Gamma(\check{M}, T^*\check{M}) \rightarrow \Gamma(\check{M}, T^*\check{M})$ is the multiplication map. The vector field X_λ vanishes at $(\mathrm{id}, e) \in \check{G} \times \check{\mathfrak{s}} \cong \check{M}$, and hence induces a Hessian endomorphism $H_{X_\lambda, (\mathrm{id}, e)} : T_{(\mathrm{id}, e)}\check{M} \rightarrow T_{(\mathrm{id}, e)}\check{M}$. Under the identification $T_{(\mathrm{id}, e)}\check{M} \cong \check{\mathfrak{g}} \times \check{\mathfrak{s}}$, one has $H_{X_\lambda, (\mathrm{id}, e)} = 0 \times E_\lambda \in \mathrm{End}(\check{\mathfrak{g}} \times \check{\mathfrak{s}})$. This description can be further simplified by replacing \check{M} with its Whittaker reduction $T^*((\check{N}, \psi) \setminus \check{G}/(\check{N}, \psi)) \cong J_{\check{G}}$. It turns out that such Hessians controlling arithmetic intersection numbers of special cycles constitute a general phenomenon for hyperspecial varieties. We plan to investigate this direction in future work.

5. EXAMPLES

In this section, we work out the constants $\epsilon_{\lambda,i}, b_\lambda$ appearing in Theorem 1.2 explicitly in some cases.

5.1. Examples of $\epsilon_{\lambda,j}$. In this section, we give examples of the number $\epsilon_{\lambda,j} \in \mathbb{Q}$ for $\lambda \in X^*(\check{T})_+$ and $1 \leq j \leq n$, where n is the rank of G .

5.1.1. Reduction to fundamental weights. The computation of the numbers $\epsilon_{\lambda,j}$ can be reduced to the computation of invariants of fundamental weights $\varpi_i \in X^*(\check{T})_+$. In fact, the computation for $\lambda_1 + \lambda_2 \in X^*(\check{T})_+$ can be reduced to $\lambda_1, \lambda_2 \in X^*(\check{T})_+$. This is provided by Proposition 5.1.

Consider $S_\lambda \in \check{\mathfrak{g}}_e^*$ defined such that for $Y \in \check{\mathfrak{g}}_{e,d_j-1}$, one has

$$S_\lambda(Y) = \langle e^{d_\lambda - d_j + 1} Y \cdot u_{\lambda,-}, u_{\lambda,+}^* \rangle. \quad (5.1)$$

Similarly, one define $T_\lambda \in \check{\mathfrak{g}}_f^*$ such that for $X \in \check{\mathfrak{g}}_{f,-d_j+1}$, one has

$$T_\lambda(X) = \left\langle \sum_{s=0}^{d_\lambda + d_j - 1} e^s X e^{d_\lambda + d_j - 1 - s} \cdot u_{\lambda,-}, u_{\lambda,+}^* \right\rangle. \quad (5.2)$$

Recall that we introduced the notation $d_\lambda = \langle 2\rho, \lambda \rangle \in \mathbb{Z}_{\geq 0}$. Define $\mathrm{deg}_\lambda := \langle e^{d_\lambda} u_{\lambda,-}, u_{\lambda,+}^* \rangle \in \mathbb{Z}_{\geq 0}$. We write $\kappa_{\lambda,j} := \kappa_\lambda|_{\check{\mathfrak{g}}_{f,-d_j+1} \otimes \check{\mathfrak{g}}_{e,d_j-1}}$.

Proposition 5.1. *For $\lambda_1, \lambda_2 \in X^*(\check{T})_+$, we have*

$$\begin{aligned} \kappa_{\lambda_1+\lambda_2, j} &= \binom{d_{\lambda_1} + d_{\lambda_2} + 1}{d_{\lambda_1} + 1} \kappa_{\lambda_1, j} \deg_{\lambda_2} + \binom{d_{\lambda_1} + d_{\lambda_2} + 1}{d_{\lambda_2} + 1} \kappa_{\lambda_2, j} \deg_{\lambda_1} \\ &+ \binom{d_{\lambda_1} + d_{\lambda_2} + 1}{d_{\lambda_1} - d_j + 1} T_{\lambda_2, j} \otimes S_{\lambda_1, j} + \binom{d_{\lambda_1} + d_{\lambda_2} + 1}{d_{\lambda_2} - d_j + 1} T_{\lambda_1, j} \otimes S_{\lambda_2, j} \end{aligned} \quad (5.3)$$

$$S_{\lambda_1+\lambda_2, j} = \binom{d_{\lambda_1} + d_{\lambda_2} - d_j + 1}{d_{\lambda_1} - d_j + 1} S_{\lambda_1, j} \deg_{\lambda_2} + \binom{d_{\lambda_1} + d_{\lambda_2} - d_j + 1}{d_{\lambda_2} - d_j + 1} S_{\lambda_2, j} \deg_{\lambda_1} \quad (5.4)$$

$$T_{\lambda_1+\lambda_2, j} = \binom{d_{\lambda_1} + d_{\lambda_2} + d_j}{d_{\lambda_1} + d_j} T_{\lambda_1, j} \deg_{\lambda_2} + \binom{d_{\lambda_1} + d_{\lambda_2} + d_j}{d_{\lambda_2} + d_j} T_{\lambda_2, j} \deg_{\lambda_1} \quad (5.5)$$

$$\deg_{\lambda_1+\lambda_2} = \binom{d_{\lambda_1} + d_{\lambda_2}}{d_{\lambda_1}} \deg_{\lambda_1} \deg_{\lambda_2}. \quad (5.6)$$

This proposition reduces the computation of κ_λ for an arbitrary $\lambda \in X^*(\check{T})$ to the computation of $\kappa_{\varpi_i}, S_{\varpi_i}, T_{\varpi_i}, \deg_{\varpi_i}$ where $\varpi_i \in X^*(\check{T})$ are the fundamental weights of \check{G} for $1 \leq i \leq n$.

5.1.2. *Case $j = 1$.* In this subsection, we derive a formula of $\epsilon_{\lambda, 1}$ for any $\lambda \in X^*(\check{T})_+$. We treat this independently since the answer in this case is particularly simple.

Proposition 5.2. *For any $\lambda \in X^*(\check{T})_+$, we have*

$$\epsilon_{\lambda, 1} = \frac{2h_G^\vee l_G d_\lambda (d_\lambda + 1)(d_\lambda + 2) \deg_\lambda}{\sum_{j=1}^n (2d_j - 2)(2d_j - 1)2d_j}. \quad (5.7)$$

Proof. Note that $\check{\mathfrak{g}}_{f, -1} = \text{Span}(f)$ and $\check{\mathfrak{g}}_{e, 1} = \text{Span}(e)$. We get

$$\epsilon_{\lambda, 1} = \kappa_\lambda(f, e) / \kappa_{\min}(f, e).$$

Consider the Lie subalgebra $\mathfrak{sl}_2 = \text{Span}(e, f, 2\rho) \subset \check{\mathfrak{g}}$. We have

$$\text{Ad}|_{\mathfrak{sl}_2} = \bigoplus_{j=1}^n \text{Sym}^{2d_j-2} \text{Std}_2.$$

Note that

$$\kappa_\lambda(f, e) = \kappa_{\text{SL}_2, \text{Sym}^{d_\lambda} \text{Std}_2}(f, e) \cdot \deg_\lambda$$

where $\kappa_{\text{SL}_2, \text{Sym}^{d_\lambda} \text{Std}_2}$ is the invariant bilinear form for SL_2 defined in (4.3) and

$$\kappa_{\min}(f, e) = \frac{1}{2h_G^\vee l_G} \kappa_{\text{Ad}}(f, e) = \frac{1}{2h_G^\vee l_G} \sum_{j=1}^n \kappa_{\text{SL}_2, \text{Sym}^{2d_j-2} \text{Std}_2}(f, e).$$

For any pair of integers $a, b \in \mathbb{Z}_{\geq 1}$, an easy computation shows

$$\frac{\kappa_{\text{SL}_2, \text{Sym}^b \text{Std}_2}}{\kappa_{\text{SL}_2, \text{Sym}^a \text{Std}_2}} = \frac{b(b+1)(b+2)}{a(a+1)(a+2)}.$$

This implies that

$$\epsilon_{\lambda, 1} = \frac{\kappa_\lambda(f, e) \deg_\lambda}{\kappa_{\min}(f, e)} = \frac{2h_G^\vee l_G d_\lambda (d_\lambda + 1)(d_\lambda + 2) \deg_\lambda}{\sum_{j=1}^n (2d_j - 2)(2d_j - 1)2d_j}.$$

□

5.1.3. *Cyclic case.* In this subsection, we illustrate a special situation where all the invariants $\epsilon_{\lambda,j}$ for different $1 \leq j \leq n$ are the same. Recall we have the representation map $r_{V_\lambda} : \check{G} \rightarrow \mathrm{GL}(V_\lambda)$ whose differential is a Lie algebra homomorphism $dr_{V_\lambda} : \check{\mathfrak{g}} \rightarrow \mathfrak{gl}(V_\lambda)$.

Proposition 5.3. *Suppose the element $dr_{V_\lambda}(e) \in \mathfrak{gl}(V_\lambda)$ is a regular nilpotent element, then we have $\epsilon_{\lambda,j} = \epsilon_{\lambda,1}$ for any $1 \leq j \leq n$.*

Proof. Under the assumption, we have $V_\lambda = \bigoplus_{s=0}^{d_\lambda} k \cdot e^s v_{\lambda,-}$. This implies that for any $X \in \check{\mathfrak{g}}_{e,d_j-1}$ and $Y \in \check{\mathfrak{g}}_{f,-d_j+1}$, we have

$$\kappa_\lambda(X, Y) = \mathrm{tr}((dr_{V_\lambda})(X) \cdot (dr_{V_\lambda})(Y)) \deg_\lambda = \kappa_{V_\lambda}(X, Y) \deg_\lambda$$

where κ_{V_λ} is defined in (4.3). This shows that $\kappa_\lambda = \frac{\kappa_{V_\lambda} \deg_\lambda}{\kappa_{\min}} \cdot \kappa_{\min}$, which implies $\epsilon_{\lambda,j} = \frac{\kappa_{V_\lambda} \deg_\lambda}{\kappa_{\min}}$ for any $1 \leq j \leq n$. \square

The assumption of Proposition 5.3 is satisfied only in the following cases:

\check{G}	Name of λ	$\epsilon_{\lambda,1}$
A_{n-1}	standard	1
C_n	standard	4
G_2	quasi-minuscule	108

For classical groups, one can write down explicit formulas for all the invariants involved in Proposition 5.1. Compared to the formulas in [Fen26], these formulas are more elementary.

5.1.4. *Type A_n .* In this subsection, we determine all the invariants in Proposition 5.1 when $G = \mathrm{PGL}_{n+1}$. In this case, we have $d_j = j + 1$ for $1 \leq j \leq n$. We have $\check{G} = \mathrm{SL}_{n+1}$. We order the fundamental weights as

$$\begin{array}{c} \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \text{---} \bullet \\ \varpi_1 \quad \varpi_2 \quad \quad \quad \varpi_{n-1} \quad \varpi_n \end{array}$$

We have $V_{\varpi_k, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^k \mathrm{Std}_{n+1, \mathbb{Z}} \in \mathrm{Rep}(\mathrm{SL}_{n+1, \mathbb{Z}})$ where $\mathrm{Std}_{n+1, \mathbb{Z}} = \bigoplus_{i=1}^{n+1} \mathbb{Z} \cdot u_i$ is the standard representation. We choose the usual maximal torus of $\mathrm{SL}_{n+1, \mathbb{Z}}$ with respect to this basis. The invariant bilinear form is given by $\kappa_{\min} = \kappa_{\mathrm{Std}_{n+1}} : \mathfrak{sl}_{n+1} \times \mathfrak{sl}_{n+1} \rightarrow k$.

Take $e \in \mathfrak{sl}_{n+1, \mathbb{Z}}$ such that

$$e \cdot u_i = \begin{cases} u_{i+1}, & 1 \leq i \leq n, \\ 0, & i = n+1. \end{cases}$$

We can choose lowest weight (co)vectors of $V_{\varpi_k, \mathbb{Z}}$ for $1 \leq k \leq n$ as

$$u_{\varpi_k, -} = u_{n-k+2} \wedge \cdots \wedge u_{n+1} \in V_{\varpi_k, \mathbb{Z}}$$

and

$$u_{\varpi_k, +}^* = (u_1 \wedge \cdots \wedge u_k)^* \in (V_{\varpi_k, \mathbb{Z}})^*,$$

where the later is given by extracting the coefficient of $u_1 \wedge \cdots \wedge u_k$ under the standard basis of $\bigwedge_{\mathbb{Z}}^k \mathrm{Std}_{n+1, \mathbb{Z}}$. With this choice, one easily checks that $\langle e^{d_{\varpi_k}} \cdot u_{\varpi_k, -}, u_{\varpi_k, +}^* \rangle > 0$.

For the computation of $S_{\varpi_k, j}, T_{\varpi_k, j}$, we choose basis of one-dimensional vector spaces $\mathfrak{sl}_{n+1, e, j}, \mathfrak{sl}_{n+1, f, -j}$ via:

$$e_j := e^{(j)} \in \mathfrak{sl}_{n+1, e, j} \tag{5.8}$$

$$f_j \in \mathfrak{sl}_{n+1, f, -j}, \quad \kappa_{\mathrm{Std}_{n+1}}(f_j, e_j) = 1 \tag{5.9}$$

where $e^{(j)}$ means the j -th power of e regarded as a $(n+1) \times (n+1)$ -matrix.

In this way, define $s_{\lambda, j} := S_\lambda(e_j) \in \mathbb{Q}$ and $t_{\lambda, j} := T_\lambda(f_j) \in \mathbb{Q}$. We have

$$\begin{aligned} s_{\varpi_k, j} &= \langle e^{s-j} e_j \cdot (u_{n-k+2} \wedge \cdots \wedge u_{n+1}), (u_1 \wedge \cdots \wedge u_k)^* \rangle \\ t_{\varpi_k, j} &= \left\langle \sum_{s=0}^{k(n+1-k)+j} e^s f_j e^{k(n+1-k)+j-s} \cdot (u_{n-k+2} \wedge \cdots \wedge u_{n+1}), (u_1 \wedge \cdots \wedge u_k)^* \right\rangle \\ \epsilon_{\varpi_k, j} &= \left\langle \sum_{s=0}^{k(n+1-k)} e^s f_j e^{k(n+1-k)-s} e_j \cdot (u_{n-k+2} \wedge \cdots \wedge u_{n+1}), (u_1 \wedge \cdots \wedge u_k)^* \right\rangle. \end{aligned}$$

Proposition 5.4. For $1 \leq k \leq n$ and $1 \leq j \leq n$, We have

$$d_{\varpi_k} = kk' \quad (5.10)$$

$$\deg_{\varpi_k} = \frac{(kk')!}{\prod_{i=1}^k \prod_{j=1}^{k'} (i+j-1)} \quad (5.11)$$

$$s_{\varpi_k, j} = \sum_{a=1}^k \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \binom{kk' - j}{k' + (\sigma(1) - 1), \dots, k' + (\sigma(a) - a) - j, \dots, k' + (\sigma(k) - k)} \quad (5.12)$$

$$t_{\varpi_k, j} = \sum_{b=1}^k \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \binom{kk' + j + 1}{k' + (\sigma(1) - 1), \dots, k' + (\sigma(b) - b) + j + 1, \dots, k' + (\sigma(k) - k)} \quad (5.13)$$

$$\epsilon_{\varpi_k, j} = \sum_{a=1}^k \sum_{b=1}^k \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \quad (5.14)$$

$$\binom{kk' + 1}{k' + (\sigma(1) - 1), \dots, k' + (\sigma(a) - a) - j, \dots, k' + (\sigma(b) - b) + j + 1, \dots, k' + (\sigma(k) - k)}$$

where $k' = n + 1 - k$. Here, the multinomial coefficient is taken to be zero if any of the entries is negative. In the formula (5.14), when $a = b$, the multinomial coefficient is understood as

$$\binom{kk' + 1}{k' + (\sigma(1) - 1), \dots, k' + (\sigma(a) - a) + 1, \dots, k' + (\sigma(k) - k)}.$$

The proof of these formulas is elementary, and we omit it.

Remark 5.5. The formula (5.14) should be compared with the [Fen26, Theorem 1.3.2].

5.1.5. *Type B_n .* In this subsection, we consider the case $G = \mathrm{SO}_{2n+1}$ in which case $d_j = 2j$ for $1 \leq j \leq n$. We have $\check{G} = \mathrm{Sp}_{2n}$. We order the fundamental weights as

$$\varpi_1 \quad \varpi_2 \quad \cdots \quad \varpi_{n-1} \quad \varpi_n$$

Define $\tilde{V}_{\varpi_k, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^k \mathrm{Std}_{2n, \mathbb{Z}} \in \mathrm{Rep}(\mathrm{Sp}_{2n, \mathbb{Z}})$ for $1 \leq k \leq n$. Here $\tilde{V}_{\varpi_1, \mathbb{Z}} = \mathrm{Std}_{2n, \mathbb{Z}} = \bigoplus_{i=1}^{2n} \mathbb{Z} \cdot u_i$ is the standard representation of $\mathrm{Sp}_{2n, \mathbb{Z}}$, and we choose the maximal torus of $\mathrm{Sp}_{2n, \mathbb{Z}}$ given by the prescribed basis of $\mathrm{Std}_{2n, \mathbb{Z}}$. The module $\mathrm{Std}_{2n, \mathbb{Z}}$ carries a symplectic form ω satisfying

$$\omega(u_i, u_j) = \begin{cases} 1, & i + j = 2n + 1, 1 \leq i \leq n, \\ 0, & i + j \neq 2n + 1. \end{cases}$$

Then $\mathrm{Sp}_{2n, \mathbb{Z}} = \mathrm{Aut}(\mathrm{Std}_{2n, \mathbb{Z}}, \omega)$. The invariant bilinear form is given by $\kappa_{\min} = \frac{1}{2} \kappa_{\mathrm{Std}_{2n}}$.

Take $e \in \mathfrak{sp}_{2n, \mathbb{Z}}$ such that

$$e \cdot u_i = \begin{cases} u_{i+1}, & 1 \leq i \leq n-1, \\ 2u_{i+1}, & i = n, \\ -u_{i+1}, & n+1 \leq i \leq 2n-1, \\ 0, & i = 2n. \end{cases}$$

We choose lowest weight (co)vectors

$$u_{\varpi_k, -} = (-1)^{k(2n-k-1)/2} u_{2n-k+1} \wedge \cdots \wedge u_{2n} \in \tilde{V}_{\varpi_k, \mathbb{Z}}$$

and

$$u_{\varpi_k, +}^* = (u_1 \wedge \cdots \wedge u_k)^* \in (\tilde{V}_{\varpi_k, \mathbb{Z}})^*$$

for $1 \leq k \leq n$. Take $V_{\varpi_k, \mathbb{Z}} \subset \tilde{V}_{\varpi_k, \mathbb{Z}}$ to be the sub-representation of $\mathrm{Sp}_{2n, \mathbb{Z}}$ generated by $u_{\varpi_k, -} \in \tilde{V}_{\varpi_k, \mathbb{Z}}$. With this choice, one easily checks that $\langle e^{d_{\varpi_k}} \cdot u_{\varpi_k, -}, u_{\varpi_k, +}^* \rangle > 0$.

We choose basis of one-dimensional vector spaces $\mathfrak{sp}_{2n, e, 2j-1}, \mathfrak{sp}_{2n, f, 1-2j}$ via:

$$e_j := e^{(2j-1)} \in \mathfrak{sp}_{2n, e, 2j-1} \quad (5.15)$$

$$f_j \in \mathfrak{sp}_{2n, f, 1-2j}, \kappa_{\text{Std}_{2n}}(f_j, e_j) = 2 \quad (5.16)$$

as in type A case. Define $s_{\lambda, j} := S_{\lambda}(e_j) \in \mathbb{Q}$ and $t_{\lambda, j} = T_{\lambda}(f_j) \in \mathbb{Q}$.

Proposition 5.6. *For $1 \leq k \leq n$ and $1 \leq j \leq n$, we have*

$$d_{\varpi_k} = k(2n - k) \quad (5.17)$$

$$\deg_{\varpi_k} = 2^k \deg_{\varpi_k}^{A_{2n-1}} \quad (5.18)$$

$$s_{\varpi_k, j} = 2^k s_{\varpi_k, 2j-1}^{A_{2n-1}} \quad (5.19)$$

$$t_{\varpi_k, j} = 2^{k+1} t_{\varpi_k, 2j-1}^{A_{2n-1}} \quad (5.20)$$

$$\epsilon_{\varpi_k, j} = 2^{k+1} \epsilon_{\varpi_k, 2j-1}^{A_{2n-1}} \quad (5.21)$$

where we add superscript A_{2n-1} to denote the corresponding invariant of type A_{2n-1} given in Proposition 5.4.

This is an immediate consequence of the corresponding result in type A given in Proposition 5.4.

5.1.6. *Type C_n .* In this subsection, we consider the case $G = \text{PSp}_{2n}$ in which case $d_j = 2j$ for $1 \leq j \leq n$. We have $\check{G} = \text{Spin}_{2n+1}$. We order the fundamental weights as

$$\bullet \xrightarrow{\varpi_1} \bullet \xrightarrow{\varpi_2} \cdots \xrightarrow{\varpi_{n-1}} \bullet \xrightarrow{\varpi_n} \bullet$$

Define $V_{\varpi_k, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^k \text{Std}_{2n+1, \mathbb{Z}}$ for $1 \leq k \leq n-1$ and $V_{2\varpi_n, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n+1, \mathbb{Z}}$. Here $V_{\varpi_1, \mathbb{Z}} = \text{Std}_{2n+1, \mathbb{Z}} = \bigoplus_{i=1}^{2n+1} \mathbb{Z} \cdot u_i$ is the standard representation of $\text{SO}_{2n+1, \mathbb{Z}}$. We choose the maximal torus of $\text{Spin}_{2n+1, \mathbb{Z}}$ determined by the basis of $\text{Std}_{2n+1, \mathbb{Z}}$. The module $\text{Std}_{2n+1, \mathbb{Z}}$ carries a quadratic form q with the corresponding symmetric bilinear form $(x, y) = q(x+y) - q(x) - q(y)$ satisfying

$$(u_i, u_j) = \begin{cases} 1, & i+j = 2n+2, i \neq n+1, \\ 2, & i=j = n+1, \\ 0, & i+j \neq 2n+2. \end{cases}$$

Then $\text{SO}_{2n+1, \mathbb{Z}} = \text{Aut}(\text{Std}_{2n+1, \mathbb{Z}}, q)^\circ$ and $\text{Spin}_{2n+1, \mathbb{Z}}$ is the universal covering group of $\text{SO}_{2n+1, \mathbb{Z}}$. The invariant bilinear form is $\kappa_{\min} = \frac{1}{4} \kappa_{\text{Std}_{2n+1}}$.

Take $e \in \mathfrak{so}_{2n+1, \mathbb{Z}}$ such that

$$e \cdot u_i = \begin{cases} 2u_{i+1}, & 1 \leq i \leq n-1, \\ u_{i+1}, & i = n, \\ -2u_{i+1}, & n+1 \leq i \leq 2n, \\ 0, & i = 2n+1. \end{cases}$$

We can choose lowest weight (co)vectors

$$u_{\varpi_k, -} = (-1)^{k(2n-k+1)/2} u_{2n-k+2} \wedge \cdots \wedge u_{2n+1} \in V_{\varpi_k, \mathbb{Z}}$$

$$u_{\varpi_k, +}^* = (u_1 \wedge \cdots \wedge u_k)^* \in (V_{\varpi_k, \mathbb{Z}})^*$$

for $1 \leq k \leq n-1$, and

$$u_{2\varpi_n, -} = (-1)^{n(n+1)/2} u_{n+2} \wedge \cdots \wedge u_{2n+1} \in V_{2\varpi_n, \mathbb{Z}}$$

$$u_{2\varpi_n, +}^* = (u_1 \wedge \cdots \wedge u_n)^* \in (V_{2\varpi_n, \mathbb{Z}})^*.$$

One can take $V_{\varpi_n, \mathbb{Z}} \in \text{Rep}(\text{Spin}_{2n+1, \mathbb{Z}})$ with lowest weight (co)vectors

$$u_{\varpi_n, -} \in V_{\varpi_n, \mathbb{Z}}, u_{\varpi_n, +}^* \in (V_{\varpi_n, \mathbb{Z}})^*$$

equipped with an injection $V_{2\varpi_n, \mathbb{Z}} \subset V_{\varpi_n, \mathbb{Z}}^{\otimes 2}$ such that

$$u_{\varpi_n, -}^{\otimes 2} = u_{2\varpi_n, -}, (u_{\varpi_n, +}^*)^{\otimes 2} = u_{2\varpi_n, +}^*.$$

One easily checks that $\langle e^{d_{\varpi_k}} \cdot u_{\varpi_k, -}, u_{\varpi_k, +}^* \rangle > 0$ for $1 \leq k \leq n$.

We choose basis of one-dimensional vector spaces $\mathfrak{so}_{2n+1, e, 2j-1}, \mathfrak{so}_{2n+1, f, 1-2j}$ via:

$$e_j := e^{(2j-1)} \in \mathfrak{so}_{2n+1, e, 2j-1} \quad (5.22)$$

$$f_j \in \mathfrak{so}_{2n+1, f, 1-2j}, \kappa_{\text{Std}_{2n+1}}(f_j, e_j) = 4. \quad (5.23)$$

Define $s_{\lambda, j} := S_\lambda(e_j) \in \mathbb{Q}$ and $t_{\lambda, j} = T_\lambda(f_j) \in \mathbb{Q}$.

Proposition 5.7. *For $1 \leq k \leq n-1$ and $1 \leq j \leq n$, we have*

$$d_{\varpi_k} = k(2n+1-k) \quad (5.24)$$

$$\deg_{\varpi_k} = 2^{k(2n-k)} \deg_{\varpi_k}^{A_{2n}} \quad (5.25)$$

$$s_{\varpi_k, j} = 2^{k(2n-k)} s_{\varpi_k, 2j-1}^{A_{2n}} \quad (5.26)$$

$$t_{\varpi_k, j} = 2^{k(2n-k)+2} t_{\varpi_k, 2j-1}^{A_{2n}} \quad (5.27)$$

$$\epsilon_{\varpi_k, j} = 2^{k(2n-k)+2} \epsilon_{\varpi_k, 2j-1}^{A_{2n}}. \quad (5.28)$$

For $k = n$ and $1 \leq j \leq n$, we have

$$d_{\varpi_n} = n(n+1)/2 \quad (5.29)$$

$$\deg_{\varpi_n} = \left(\frac{2d_{\varpi_n}}{d_{\varpi_n}} \right)^{-1} 2^{n^2} \deg_{\varpi_n}^{A_{2n}})^{1/2} \quad (5.30)$$

$$s_{\varpi_n, j} = \left(\frac{2d_{\varpi_n} - 2j + 1}{d_{\varpi_n}} \right)^{-1} \deg_{\varpi_n}^{-1} 2^{n^2-1} s_{\varpi_n, 2j-1}^{A_{2n}} \quad (5.31)$$

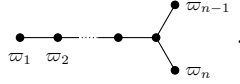
$$t_{\varpi_n, j} = \left(\frac{2d_{\varpi_n} + 2j}{d_{\varpi_n}} \right)^{-1} \deg_{\varpi_n}^{-1} 2^{n^2+1} t_{\varpi_n, 2j-1}^{A_{2n}} \quad (5.32)$$

$$\epsilon_{\varpi_n, j} = \left(\frac{2d_{\varpi_n} + 1}{d_{\varpi_n}} \right)^{-1} \deg_{\varpi_n}^{-1} (2^{n^2+1} \epsilon_{\varpi_n, 2j-1}^{A_{2n}} - \left(\frac{2d_{\varpi_n} + 1}{d_{\varpi_n} - 2j + 1} \right) s_{\varpi_n, j} t_{\varpi_n, j}) \quad (5.33)$$

This follows immediately from Proposition 5.4 and Proposition 5.1.

Remark 5.8. The formula (5.33) should be compared with the [Fen26, Theorem 1.3.6].

5.1.7. *Type D_n .* In this subsection, we consider the case $G = \text{PSO}_{2n}$ in which case we arrange $d_j = 2j$ for $1 \leq j \leq n-1$ and $d_n = n$. Note that when n is even, we have $d_{n/2} = d_n = n$. We have $\check{G} = \text{Spin}_{2n}$ whose fundamental weights are ordered as



Define $V_{\varpi_k, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^k \text{Std}_{2n, \mathbb{Z}} \in \text{Rep}(\text{Spin}_{2n, \mathbb{Z}})$ for $1 \leq k \leq n-2$ and $V_{\varpi_{n-1} + \varpi_n, \mathbb{Z}} = \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n, \mathbb{Z}} \in \text{Rep}(\text{Spin}_{2n, \mathbb{Z}})$. Here $V_{\varpi_1, \mathbb{Z}} = \text{Std}_{2n, \mathbb{Z}} = \bigoplus_{i=1}^{2n} k \cdot u_i$ is the standard representation of SO_{2n} . We choose the maximal torus of $\text{Spin}_{2n, \mathbb{Z}}$ determined by the basis of $\text{Std}_{2n, \mathbb{Z}}$. The module $\text{Std}_{2n, \mathbb{Z}}$ carries a quadratic form q with the corresponding symmetric bilinear form $(x, y) = q(x+y) - q(x) - q(y)$ satisfying

$$(u_i, u_j) = \begin{cases} 1, & i+j = 2n+1, \\ 0, & i+j \neq 2n+1. \end{cases}$$

Then $\text{SO}_{2n, \mathbb{Z}} = \text{Aut}(\text{Std}_{2n, \mathbb{Z}}, q)^\circ$ and $\text{Spin}_{2n, \mathbb{Z}}$ is the universal covering group of $\text{SO}_{2n, \mathbb{Z}}$. The invariant bilinear form on \mathfrak{so}_{2n} in given by $\kappa_{\min} = \frac{1}{2} \kappa_{\text{Std}_{2n}}$.

Take $e \in \mathfrak{so}_{2n, \mathbb{Z}}$ such that

$$e \cdot u_i = \begin{cases} u_{i+1}, & 1 \leq i \leq n-2, \\ u_n + u_{n+1}, & i = n-1, \\ -u_{n+2}, & i = n, \\ -u_{i+1}, & n+1 \leq i \leq 2n-1, \\ 0, & i = 2n. \end{cases}$$

We can choose lowest weight (co)vectors

$$u_{\varpi_k, -} = (-1)^{k(2n-k-1)/2} u_{2n-k+1} \wedge \cdots \wedge u_{2n} \in V_{\varpi_k, \mathbb{Z}}$$

$$u_{\varpi_k,+}^* = (u_1 \wedge \cdots \wedge u_k)^* \in (V_{\varpi_k,\mathbb{Z}})^*$$

for $1 \leq k \leq n-2$, and

$$u_{\varpi_{n-1}+\varpi_n,-} = (-1)^{n(n-1)/2} u_{n+2} \wedge \cdots \wedge u_{2n} \in V_{\varpi_{n-1}+\varpi_n,\mathbb{Z}}$$

$$u_{\varpi_{n-1}+\varpi_n,+}^* = (u_1 \wedge \cdots \wedge u_{n-1})^* \in (V_{\varpi_{n-1}+\varpi_n,\mathbb{Z}})^*.$$

Consider elements

$$u_{2\varpi_{n-1},-} = (-1)^{n(n-1)/2} u_{n+1} \wedge \cdots \wedge u_{2n} \in \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}$$

$$u_{2\varpi_{n-1},+}^* = (u_1 \wedge \cdots \wedge u_n)^* \in \left(\bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}\right)^*$$

$$u_{2\varpi_n,-} = (-1)^{n(n-1)/2} u_n \wedge u_{n+2} \wedge \cdots \wedge u_{2n} \in \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}$$

$$u_{2\varpi_n,+}^* = (u_1 \wedge \cdots \wedge u_{n-1} \wedge u_n)^* \in \left(\bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}\right)^*.$$

Let $V_{2\varpi_{n-1},\mathbb{Z}}, V_{2\varpi_n,\mathbb{Z}} \subset \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}$ be the sub-representation of $\text{Spin}_{2n,\mathbb{Z}}$ generated by $u_{2\varpi_{n-1},-}, u_{2\varpi_n,-} \in \bigwedge_{\mathbb{Z}}^n \text{Std}_{2n,\mathbb{Z}}$, respectively. With these choices, one can take $V_{\varpi_{n-1},\mathbb{Z}}, V_{\varpi_n,\mathbb{Z}} \in \text{Rep}(\text{Spin}_{2n,\mathbb{Z}})$ with lowest weight (co)vectors

$$u_{\varpi_{n-1},-} \in V_{\varpi_{n-1},\mathbb{Z}}, u_{\varpi_{n-1},+}^* \in (V_{\varpi_{n-1},\mathbb{Z}})^*$$

$$u_{\varpi_n,-} \in V_{\varpi_n,\mathbb{Z}}, u_{\varpi_n,+}^* \in (V_{\varpi_n,\mathbb{Z}})^*$$

such that there exists embeddings $V_{2\varpi_{n-1},\mathbb{Z}} \subset V_{\varpi_{n-1},\mathbb{Z}}^{\otimes 2}, V_{2\varpi_n,\mathbb{Z}} \subset V_{\varpi_n,\mathbb{Z}}^{\otimes 2}$ such that

$$u_{\varpi_{n-1},-}^{\otimes 2} = u_{2\varpi_{n-1},-}, (u_{\varpi_{n-1},+}^*)^{\otimes 2} = u_{2\varpi_{n-1},+}, u_{\varpi_n,-}^{\otimes 2} = u_{2\varpi_n,-}, (u_{\varpi_n,+}^*)^{\otimes 2} = u_{2\varpi_n,+}.$$

Moreover, there exists embedding $V_{\varpi_{n-1}+\varpi_n,\mathbb{Z}} \subset V_{\varpi_{n-1},\mathbb{Z}} \otimes V_{\varpi_n,\mathbb{Z}}$ such that

$$u_{\varpi_{n-1},-} \otimes u_{\varpi_n,-} = u_{\varpi_{n-1}+\varpi_n,-}, u_{\varpi_{n-1},+}^* \otimes u_{\varpi_n,+}^* = u_{\varpi_{n-1}+\varpi_n,+}^*.$$

With these choices, one easily checks that $\langle e^{d_{\varpi_k}} \cdot u_{\varpi_k,-}, u_{\varpi_k,+}^* \rangle > 0$ for $1 \leq k \leq n$.

For $1 \leq j \leq n-1$, define $e_j \in \mathfrak{so}_{2n,e,2j-1}, f_j \in \mathfrak{so}_{2n,f,1-2j}$ via:

$$e_j := e^{(2j-1)} \in \mathfrak{so}_{2n,e,2j-1} \tag{5.34}$$

$$f_j \in \text{Span}(f^{(2j-1)}) \subset \mathfrak{so}_{2n,f,1-2j}, \kappa_{\text{Std}_{2n}}(f_j, e_j) = 2. \tag{5.35}$$

For $j = n$, take $e_n \in \mathfrak{so}_{2n,e,n-1}$ such that

$$e_n \cdot u_i = \begin{cases} u_n - u_{n+1}, & i = 1, \\ u_{2n}, & i = n, \\ -u_{2n}, & i = n+1, \\ 0, & \text{else} \end{cases}$$

and $f_n \in \mathfrak{so}_{2n,f,1-n}$ such that

$$f_n \cdot u_i = \begin{cases} u_1/2, & i = n, \\ -u_1/2, & i = n+1, \\ (u_n - u_{n+1})/2, & i = 2n, \\ 0, & \text{else.} \end{cases}$$

Then one has $\kappa_{\text{Std}_{2n}}(f_n, e_n) = 2$. Moreover, when n is even, one has $\kappa_{\text{Std}_{2n}}(f_n, e_{n/2}) = \kappa_{\text{Std}_{2n}}(f_{n/2}, e_n) = 0$. Define $s_{\lambda,j} := S_{\lambda}(e_j) \in \mathbb{Q}$ and $t_{\lambda,j} := T_{\lambda}(f_j) \in \mathbb{Q}$.

Take $\sigma \in \text{Aut}(\text{Spin}_{2n,\mathbb{Z}})$ to be the unique non-trivial outer automorphism fixing the pinning determined by $e \in \mathfrak{so}_{2n,\mathbb{Z}}$ and the first vertex of the Dynkin diagram. We have $\sigma(e_j) = e_j, \sigma(f_j) = f_j$ for $1 \leq j \leq n-1$, $\sigma(e_n) = -e_n$, and $\sigma(f_n) = -f_n$.

When n is odd, we have $\epsilon_{\varpi_k, j} = \kappa_{\varpi_k}(e_j, f_j)$ for all j, k . When n is even, the same formula holds for $j \neq n/2, n$ or $k \neq n-1, n$ while the numbers $\epsilon_{\lambda, n/2}$ and $\epsilon_{\lambda, n}$ for $\lambda = \varpi_{n-1}, \varpi_n$ are eigenvalues of the 2×2 -matrix

$$\begin{pmatrix} \kappa_{\lambda}(f_{n/2}, e_{n/2}) & \kappa_{\lambda}(f_n, e_{n/2}) \\ \kappa_{\lambda}(f_{n/2}, e_n) & \kappa_{\lambda}(f_n, e_n) \end{pmatrix}. \quad (5.36)$$

Proposition 5.9. *For $1 \leq k \leq n-2$ and $1 \leq j \leq n-1$, we have*

$$d_{\varpi_k} = k(2n-1-k), \quad (5.37)$$

$$\deg_{\varpi_k} = 2^k \deg_{\varpi_k}^{A_{2n-2}}, \quad (5.38)$$

$$s_{\varpi_k, j} = 2^k s_{\varpi_k, 2j-1}^{A_{2n-2}}, \quad (5.39)$$

$$t_{\varpi_k, j} = 2^{k+1} t_{\varpi_k, 2j-1}^{A_{2n-2}}, \quad (5.40)$$

$$\epsilon_{\varpi_k, j} = 2^{k+1} \epsilon_{\varpi_k, 2j-1}^{A_{2n-2}}. \quad (5.41)$$

For $1 \leq k \leq n-2$ and $j = n$, we have

$$s_{\varpi_k, n} = 0, \quad (5.42)$$

$$t_{\varpi_k, n} = 0, \quad (5.43)$$

$$\epsilon_{\varpi_k, n} = 2^k \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \begin{pmatrix} k(2n-1-k) + 1 \\ 2n-k + (\sigma(1)-1), 2n-k-1 + (\sigma(2)-2), \dots, 2n-k-1 + (\sigma(k)-k) \end{pmatrix}. \quad (5.44)$$

For $k = n-1, n$ and $1 \leq j \leq n-1$, we have

$$d_{\varpi_k} = n(n-1)/2, \quad (5.45)$$

$$\deg_{\varpi_k} = \left(\frac{2d_{\varpi_{n-1}}}{d_{\varpi_{n-1}}} \right)^{-1} 2^{n-1} \deg_{\varpi_{n-1}}^{A_{2n-2}})^{1/2}, \quad (5.46)$$

$$s_{\varpi_k, j} = \left(\frac{2d_{\varpi_{n-1}} - 2j + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} 2^{n-2} s_{\varpi_{n-1}, 2j-1}^{A_{2n-2}}, \quad (5.47)$$

$$t_{\varpi_k, j} = \left(\frac{2d_{\varpi_{n-1}} + 2j}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} 2^{n-1} t_{\varpi_{n-1}, 2j-1}^{A_{2n-2}}, \quad (5.48)$$

$$\kappa_{\varpi_k}(f_j, e_j) = \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} (2^{n-1} \epsilon_{\varpi_{n-1}, 2j-1}^{A_{2n-2}} - \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}} - 2j + 1} \right) s_{\varpi_{n-1}, j} t_{\varpi_{n-1}, j}). \quad (5.49)$$

For $k = n-1, n$ and $j = n$, we have

$$s_{\varpi_{n-1}, n} = -s_{\varpi_n, n} = (-1)^{n-1} \left(\frac{2d_{\varpi_{n-1}} - n + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} 2^{n-2} \deg_{\varpi_{n-1}}^{A_{2n-3}}, \quad (5.50)$$

$$t_{\varpi_{n-1}, n} = -t_{\varpi_n, n} = \left(\frac{2d_{\varpi_{n-1}} + n}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} 2^{n-2} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \quad (5.51)$$

$$\begin{pmatrix} n^2 \\ n-1 + (\sigma(1)-1), \dots, n-1 + (\sigma(n-1) - (n-1)), n-1 + \sigma(n) \end{pmatrix} \\ \kappa_{\varpi_k}(f_j, e_j) = \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} (2^{-1} \epsilon_{\varpi_{n-1} + \varpi_n, n} + \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}} - n + 1} \right) s_{\varpi_{n-1}, j} t_{\varpi_{n-1}, j}) \quad (5.52)$$

where

$$\epsilon_{\varpi_{n-1} + \varpi_n, n} = 2^{n-1} \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) \begin{pmatrix} n(n-1) + 1 \\ n+1 + (\sigma(1)-1), n + (\sigma(2)-2), \dots, n + (\sigma(n-1) - (n-1)) \end{pmatrix}. \quad (5.53)$$

Finally, when n is even, $k = n-1, n$, we have

$$\begin{aligned} \kappa_{\varpi_{n-1}}(f_n, e_{n/2}) &= -\kappa_{\varpi_n}(f_n, e_{n/2}) = \\ & \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} (2^{-1} \kappa_{2\varpi_{n-1}}(f_n, e_{n/2}) - \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}} + 1 - n} \right) t_{\varpi_{n-1}, n} s_{\varpi_{n-1}, n/2}) \end{aligned} \quad (5.54)$$

$$\begin{aligned} \kappa_{\varpi_{n-1}}(f_{n/2}, e_n) &= -\kappa_{\varpi_n}(f_{n/2}, e_n) = \\ \left(\frac{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}}} \right)^{-1} \deg_{\varpi_{n-1}}^{-1} (2^{-1} \kappa_{2\varpi_{n-1}}(f_{n/2}, e_n) - \binom{2d_{\varpi_{n-1}} + 1}{d_{\varpi_{n-1}} + 1 - n}) t_{\varpi_{n-1}, n/2} s_{\varpi_{n-1}, n} \end{aligned} \quad (5.55)$$

where

$$\kappa_{2\varpi_{n-1}}(f_n, e_{n/2}) = 2^{n-1} \sum_{a=1}^{n-1} \sum_{\sigma \in S_n} \text{sgn}(\sigma) \cdot \quad (5.56)$$

$$\begin{aligned} &\binom{n(n-1)+1}{n-1+(\sigma(1)-1), \dots, \sigma(a)-a, \dots, n-1+(\sigma(n-1)-(n-1)), n-1+\sigma(n)} \\ \kappa_{2\varpi_{n-1}}(f_{n/2}, e_n) &= -2^n t_{\varpi_{n-1}, n-1}^{A_{2n-3}}. \end{aligned} \quad (5.57)$$

When $j \neq n$, these formulas follow immediately from Proposition 5.4 and Proposition 5.1. The case $j = n$ can be treated similarly.

Remark 5.10. The formulas in Proposition 5.9 can be compared with the formulas in [Fen26, Theorem 1.3.8].

5.1.8. *Small rank cases.* In this subsection, we list the invariants $\epsilon_{\varpi_i, j}$ for all semisimple simply-connected groups \check{G} of classical types with rank not exceeding 4. These invariants together with the relevant invariants d_λ, \deg_λ are given in Table 1, Table 2, Table 3, Table 4.

TABLE 1. Value of $\deg_\lambda, d_\lambda, \epsilon_{\lambda,1}$ for classical groups of rank 1

\check{G}	λ	\deg_λ	d_λ	$\epsilon_{\lambda,1}$
A_1	ϖ_1	1	1	1

TABLE 2. Value of $\deg_\lambda, d_\lambda, (\epsilon_{\lambda,j})_{j=1}^2$ for classical groups of rank 2

\check{G}	λ	\deg_λ	d_λ	$(\epsilon_{\lambda,j})_{j=1}^2$
A_2	ϖ_1	1	2	(1, 1)
A_2	ϖ_2	1	2	(1, 1)
B_2	ϖ_1	8	4	(32, 32)
B_2	ϖ_2	2	3	(4, 4)

TABLE 3. Value of $\deg_\lambda, d_\lambda, (\epsilon_{\lambda,j})_{j=1}^3$ for classical groups of rank 3

\check{G}	λ	\deg_λ	d_λ	$(\epsilon_{\lambda,j})_{j=1}^3$
A_3	ϖ_1	1	3	(1, 1, 1)
A_3	ϖ_2	2	4	(4, 2, 4)
A_3	ϖ_3	1	3	(1, 1, 1)
B_3	ϖ_1	32	6	(128, 128, 128)
B_3	ϖ_2	10752	10	(168960, 67584, 125952)
B_3	ϖ_3	16	6	(64, 40, 64)
C_3	ϖ_1	2	5	(4, 4, 4)
C_3	ϖ_2	56	8	(384, 184, 328)
C_3	ϖ_3	336	9	(3168, 1568, 2720)

TABLE 4. Value of $\deg_\lambda, d_\lambda, (\epsilon_{\lambda,j})_{j=1}^4$ for classical groups of rank 4

\check{G}	λ	\deg_λ	d_λ	$(\epsilon_{\lambda,j})_{j=1}^4$
A_4	ϖ_1	1	4	(1, 1, 1, 1)
A_4	ϖ_2	5	6	(14, 6, 9, 13)
A_4	ϖ_3	5	6	(14, 6, 9, 13)
A_4	ϖ_4	1	4	(1, 1, 1, 1)
B_4	ϖ_1	128	8	(512, 512, 512, 512)
B_4	ϖ_2	1757184	14	(132800768, 11501568, 19169280, 23216128)
B_4	ϖ_3	2867724288	18	(108973522944, 26417823744, 42420928512, 65536524288)
B_4	ϖ_4	768	10	(5632, 2432, 3584, 5248)
C_4	ϖ_1	2	7	(4, 4, 4, 4)
C_4	ϖ_2	528	12	(4576, 1664, 3008, 3424)
C_4	ϖ_3	48048	15	(777920, 232128, 349632, 555968)
C_4	ϖ_4	384384	16	(7468032, 2376192, 3355392, 5392128)
D_4	ϖ_1	2	6	(4, 4, 4, 2)
D_4	ϖ_2	168	10	(1320, 528, 984, 528)
D_4	ϖ_3	2	6	(4, 4, 4, 2)
D_4	ϖ_4	2	6	(4, 4, 4, 2)

5.1.9. *Exceptional types.* In this subsection, we consider the case that G is of exceptional type. Using Sagemath [The25], one can calculate the numbers $(\epsilon_{\lambda,j})_{j=1}^n$ when $\lambda \in X^*(\check{T})_+$ is the (quasi-)minuscule or adjoint weight. Table 5 gives the numbers d_λ, \deg_λ and Table 6 gives the numbers $(\epsilon_{\lambda,j})_{j=1}^n$.

Remark 5.11. Our computation suggests that $\epsilon_{\lambda,i}$ are always algebraic integers. Moreover, when G is not of type D_n for $n \geq 4$ even, they are always positive integers. We do not have a conceptual explanation for this phenomenon.

TABLE 5. Value of \deg_λ, d_λ for exceptional types

\check{G}	Name of λ	\deg_λ	d_λ
G_2	quasi-minuscule	18	6
G_2	adjoint	13608	10
F_4	quasi-minuscule	4992	16
F_4	adjoint	154791936	22
E_6	minuscule	78	16
E_6	adjoint	151164	22
E_7	minuscule	13110	27
E_7	adjoint	141430680	34
E_8	adjoint	126937516885200	58

5.2. **Examples of b_λ .** In this section, we give examples of the number $b_\lambda \in \mathbb{Q}$ for $\lambda \in X^*(\check{T})_+$.

5.2.1. *Reduction to fundamental weights.* The calculation of the numbers b_λ can be reduced to the calculation of a sum of two fundamental weights $\varpi_i + \varpi_j \in X^*(\check{T})_+$. In fact, the calculation for $\lambda_1 + \lambda_2 + \lambda_3 \in X^*(\check{T})_+$ can be reduced to the calculation of $\lambda_k, \lambda_i + \lambda_j \in X^*(\check{T})_+$ for $1 \leq i, j, k \leq 3$. This reduction procedure is provided by Proposition 5.12.

TABLE 6. Values of $\epsilon_{\lambda,j}$ for exceptional types

\check{G}	Name of λ	$(\epsilon_{\lambda,j})_{j=1}^n$
G_2	quasi-minuscule	(108, 108)
G_2	adjoint	(320760, 239112)
F_4	quasi-minuscule	(52224, 27648, 32640, 47232)
F_4	adjoint	(4016652288, 1610956800, 2099613696, 2908827648)
E_6	minuscule	(408, 221, 216, 255, 299, 369)
E_6	adjoint	(1961256, 490314, 786600, 1025202, 967518, 1420326)
E_7	minuscule	(120060, 58320, 48024, 69920, 67176, 85376, 101744)
E_7	adjoint	(2530864800, 643719960, 1025744616, 921331160, 1301226984, 1386811784, 1807586936)
E_8	adjoint	(3503065990170600, 1035140220518880, 1116031452545520, 1289228099378520, 1602802318771080, 1661879186158800, 2045534982573600, 2471069708566200)

Proposition 5.12. For $\lambda_1, \lambda_2, \lambda_3 \in X^*(\check{T})_+$, we have

$$\begin{aligned}
& b_{\lambda_1+\lambda_2+\lambda_3} = \\
& \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_1} \end{matrix} \right) \deg_{\lambda_1} b_{\lambda_2+\lambda_3} + \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_2} \end{matrix} \right) \deg_{\lambda_2} b_{\lambda_1+\lambda_3} \\
& + \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_3} \end{matrix} \right) \deg_{\lambda_3} b_{\lambda_1+\lambda_2} - \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_1} + 1 \end{matrix} \right) b_{\lambda_1} \deg_{\lambda_2+\lambda_3} \\
& - \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_2} + 1 \end{matrix} \right) b_{\lambda_2} \deg_{\lambda_1+\lambda_3} - \left(\begin{matrix} d_{\lambda_1} + d_{\lambda_2} + d_{\lambda_3} + 1 \\ d_{\lambda_3} + 1 \end{matrix} \right) b_{\lambda_3} \deg_{\lambda_1+\lambda_2}
\end{aligned} \tag{5.58}$$

This follows immediately from Theorem 4.4.

5.2.2. *Minuscule case.* When $\lambda \in X^*(\check{T})_+$ is minuscule, the number b_λ is particularly easy to determine:

Proposition 5.13. When $\lambda \in X^*(\check{T})_+$ is minuscule, we have

$$b_\lambda = \frac{1}{2} \kappa_{\min}(\lambda, \lambda) (d_\lambda + 1) \deg_\lambda.$$

This follows immediately from Theorem 4.4. The following is a table of the invariants involved in the formula of b_λ for all minuscule weights $\lambda \in X^*(\check{T})$:

\check{G}	Name of λ	$\kappa_{\min}(\lambda, \lambda)$	d_λ	\deg_λ
$A_n (n \geq 1)$	$\varpi_k (1 \leq k \leq n)$	$k(n+1-k)/(n+1)$	$k(n+1-k)$	(5.11)
$B_n (n \geq 1)$	spin	$n/2$	$n(n+1)/2$	(5.30)
$C_n (n \geq 1)$	standard	1	$2n-1$	2
$D_n (n \geq 3)$	standard	1	$2n-2$	2
$D_n (n \geq 3)$	half-spin	$n/4$	$n(n-1)/2$	(5.46)
E_6	minuscule	$4/3$	16	78
E_7	minuscule	$3/2$	27	13110

5.2.3. *Classical types.* When \check{G} is of classical type, one can write down formulas parallel to those in §5.1.4, §5.1.5, §5.1.6, and §5.1.7. We omit the details.

5.2.4. *Exceptional types.* When \check{G} is of exceptional type, with the help of Sagemath [The25], we calculate the number b_λ when $\lambda \in X^*(\check{T})_+$ is the (quasi-)minuscule or adjoint weight. The result is collected in Table 7.

TABLE 7. Values of b_λ for exceptional types

\check{G}	Name of λ	b_λ
G_2	quasi-minuscule	108
G_2	adjoint	279936
F_4	quasi-minuscule	79872
F_4	adjoint	5318025216
E_6	minuscule	884
E_6	adjoint	3325608
E_7	minuscule	275310
E_7	adjoint	4808643120
E_8	adjoint	7362375979341600

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA

Email address: wangzeyu@mit.edu

UNIVERSITY OF CALIFORNIA BERKELEY, DEPARTMENT OF MATHEMATICS, BERKELEY, CA 94720, USA

Email address: weiqing@berkeley.edu