

ORTHOGONAL GAN–GROSS–PRASAD CONJECTURE IN GEOMETRIC LANGLANDS

KAIYI CHEN, ZEYU WANG

ABSTRACT. We prove part of the global Gan–Gross–Prasad conjecture for $G = \mathrm{SO}_n \times \mathrm{SO}_{n-1}$ and $H = \mathrm{SO}_{n-1}$ in the unramified de Rham geometric Langlands setting proposed by Ben-Zvi–Sakellaridis–Venkatesh.

CONTENTS

1. Introduction	1
2. Local constancy	4
3. Classical limit	11
4. Period of differential operators	16
References	20

1. INTRODUCTION

1.1. Setup.

1.1.1. *Automorphic side.* Let C be a connected smooth projective curve over $\mathbb{C}$. Let $H = \mathrm{SO}_{n-1}$ and $H' = \mathrm{SO}_n$ with $n \geq 4$, and let $G = H' \times H$. Let $X = G/H$ be the corresponding spherical G -variety. We use $T \subset B = NT \subset G$ to denote the subgroups as usual. Let

$$\mathrm{Bun}_G := \mathrm{Map}(C, \mathbb{B}G)$$

be the moduli stack of G -bundles on C , and let

$$\mathrm{Bun}_G^X := \mathrm{Bun}_H = \mathrm{Map}(C, [X/G]).$$

There is a natural map

$$\pi_X : \mathrm{Bun}_G^X \rightarrow \mathrm{Bun}_G.$$

Following [BZSV24, §10.3], one can define the (unnormalized) *period sheaf*

$$\mathcal{P}_X := (\pi_{X, \mathrm{dR}})_! \mathbb{C}_{\mathrm{Bun}_G^X} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \tag{1.1}$$

where $\mathbb{C}_{\mathrm{Bun}_G^X}$ is the constant D -module on Bun_G^X and $(\pi_{X, \mathrm{dR}})_!$ is the de Rham pushforward.

1.1.2. *Spectral side.* Let $\check{G} = \check{H}' \times \check{H}$ be the Langlands dual group of G . Let $\check{M} \in \mathrm{Rep}(\check{G})$ be the tensor product of the standard representations of $\check{H}$ and $\check{H}'$. Let

$$\mathrm{Loc}_{\check{G}} := \mathrm{Map}(C_{\mathrm{dR}}, \mathbb{B}\check{G})$$

be the moduli stack of de Rham $\check{G}$ -local systems on C .

Let $\check{M}_{\mathrm{univ}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}) \otimes \mathrm{D}\text{-mod}(C)$ be the universal $\check{G}$ -local system on C associated to the representation $\check{M}$. We use a normalization in which $\check{M}_{\mathrm{univ}}$ lies in cohomological degree -2 . Its de Rham cohomology defines

$$\Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*) \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}).$$

This is a derived vector bundle on $\mathrm{Loc}_{\check{G}}$ of (virtual) rank $-(2g-2) \dim(\check{M})$. Since $\check{M}$ is a symplectic self-dual representation of $\check{G}$, the de Rham cohomology group $\Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)[-1]$ carries a symmetric bilinear form.

Let $\mathrm{Loc}_{\check{G}}^{\circ} \subset \mathrm{Loc}_{\check{G}}$ be the open substack where $\Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)[-1]$ is concentrated in degree 0. Following [BZSV24, §12.8], one considers the Clifford algebra

$$\mathbb{O}_{\check{M}}^{\circ} := \mathrm{Cl}(\Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)[-1]) \in \mathrm{Alg}(\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\circ})).$$

1.1.3. *Comparing the two sides.* The series of papers [GR24a, ABC⁺24a, CCF⁺24, ABC⁺24b, GR24b] establishes an equivalence of categories

$$\mathbb{L}_G : \mathrm{D}\text{-mod}(\mathrm{Bun}_G) \xrightarrow{\sim} \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}).$$

The following conjecture is [BZSV24, Conjecture 12.8.1] in our setting:

Conjecture 1.1. *The object $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_{\check{G}}^{\circ}}$ is an irreducible $\mathbb{O}_{\check{M}}^{\circ}$ -module in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}}^{\circ})$, that is, there is a canonical isomorphism*

$$\mathbb{O}_{\check{M}}^{\circ} \xrightarrow{\sim} \underline{\mathrm{End}}_{\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\circ})}(\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_{\check{G}}^{\circ}}) \in \mathrm{Alg}(\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\circ})).$$

1.2. **Main result.** Let

$$j_{\mathrm{irr}} : \mathrm{Loc}_{\check{G}}^{\mathrm{irr}} \hookrightarrow \mathrm{Loc}_{\check{G}}$$

be the locus of irreducible $\check{G}$ -local systems. One has $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}} \subset \mathrm{Loc}_{\check{G}}^{\circ}$. Note that $\pi_0(\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}) = \pi_0(\mathrm{Loc}_{\check{G}}) = \pi_1(\check{G}) \cong \mathbb{Z}/2\mathbb{Z}$. Let

$$\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0} \subset \mathrm{Loc}_{\check{G}}^{\mathrm{irr}}$$

be the neutral component. The main result of this paper is the following:

Theorem 1.2. *Conjecture 1.1 holds when restricted to $\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}$.*

Remark 1.3. Our proof involves computations for the sheaf of twisted differential operators on Bun_G at the critical level, which corresponds to the pushforward of the structure sheaf along the map $\mathrm{Op}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}$ from the moduli stack of $\check{G}$ -opers. The image of this map lies in the neutral component $\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}$, and this is the reason we only prove Conjecture 1.1 for $\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}$. However, the remaining component of $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}$ can be addressed in a similar manner, by considering Kac–Moody localizations of other modules rather than the vacuum, which corresponds to considering other monodromy-free opers instead of the regular opers. Also, the restricted variation variant of Theorem 1.2 in positive characteristic should follow using the techniques of [GR25]. Combining this result with the results of [LW25, Wan25, Wan26], one can deduce a higher-derivative function-field version of the arithmetic Gan–Gross–Prasad conjecture in the unramified setting in the spirit of [YZ17]. We plan to address these in a future version of this article.

Remark 1.4. When $g < 2$, the irreducible locus $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}$ is empty, and the statement of Theorem 1.2 is vacuous. So we assume $g \geq 2$ throughout this paper.

1.3. **Strategy of proof.** The construction in [LW25] equips $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}}$ with a natural action of the Clifford algebra $\mathbb{O}_{\check{M}}^{\circ}|_{\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}}$ (see Proposition 2.8). It therefore suffices to prove the following:

Theorem 1.5. *The object*

$$\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}}[\delta_H - \delta_{N_{\Omega_C^{1/2}}}] \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0})$$

is a locally free sheaf in $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\mathrm{irr},0})^{\vee}$, with rank equal to $2^{(g-1)\dim(\check{M})}$. Here

- $\delta_H = \dim \mathrm{Bun}_H$.
- $\delta_{N_{\Omega_C^{1/2}}} = \dim \mathrm{Bun}_{N_{\Omega_C^{1/2}}}$ where $\mathrm{Bun}_{N_{\Omega_C^{1/2}}} := \mathrm{Map}(C, \mathbb{B}(N \rtimes \mathbb{G}_m)) \times_{\mathrm{Map}(C, \mathbb{B}\mathbb{G}_m)} \{\Omega_C^{1/2}\}$. Here $\mathbb{G}_m$ acts on N by $2\rho \in X_*(T)$.

We prove the following two propositions, which together imply Theorem 1.5:

Proposition 1.6 (Proposition 2.1). *The sheaf $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}}^{\mathrm{irr}})$ has bounded cohomological degrees with locally free cohomologies.*

Proposition 1.7 (Proposition 4.1). *There exists $\sigma \in \mathrm{Loc}_{\check{G}}^{\mathrm{irr},0}(\mathbb{C})$ such that the fiber $i_{\sigma}^*(\mathbb{L}_G(\mathcal{P}_X))[\delta_H - \delta_{N_{\Omega_C^{1/2}}}] \in \mathrm{Vect}$ lies in $\mathrm{Vect}^{\vee}$ and has dimension $2^{(g-1)\dim(\check{M})}$, where $i_{\sigma} : \{\sigma\} \rightarrow \mathrm{Loc}_{\check{G}}$ is the inclusion.*

We will prove Proposition 1.6 in §2 by showing that $\mathbb{L}_G(\mathcal{P}_X)$ has vanishing Atiyah class locally on $\mathrm{Loc}_G^{\mathrm{irr}}$ (Proposition 2.6), which will be a consequence of the Plancherel algebra computation in [BFT22] (Corollary 2.12). We will prove Proposition 1.7 in §4 by computing the period integral of Hecke eigensheaves arising from the quantization of the Hitchin system in [BD91]. For this computation, we use a naturally defined filtration and reduce the problem to a classical limit calculation in §3. This is similar to the strategy used in the proof of the index formula in [FR25, Theorem 6.1.2.1] (see Remark 4.7 for a more detailed comparison).

1.4. Notations. In this article, we fix a smooth projective connected curve C over the complex numbers with genus $g \geq 2$. We will fix $G, H, H', \check{G}$ as in §1 and use $\mathfrak{g}, \mathfrak{h}, \mathfrak{h}', \check{\mathfrak{g}}$ to denote their Lie algebras. It should be clear from the context that some of our results are valid for more general groups.

By default, work with presentable ∞ -categories. Given such a category $\mathcal{C}$ and objects x, y , we write $\mathrm{Hom}^0(x, y) := \pi_0(\mathrm{Map}_{\mathcal{C}}(x, y))$.

We use Bun_G to denote the moduli stack of G -bundles. For each finite-dimensional representation V of G , we denote by $V_{\mathrm{univ}} \in \mathrm{QCoh}(\mathrm{Bun}_G \times C)$ the universal vector bundle associated to V . We use

$$\mathcal{L}_{G,V} := \det(\Gamma(C, V_{\mathrm{univ}})) \otimes \det(\Gamma(C, \mathcal{O}_C^{\oplus \dim(V)}))^{-1} \in \mathrm{Pic}(\mathrm{Bun}_G)$$

to denote the line bundle on Bun_G associated to V .

We use $\mathrm{Loc}_{\check{G}}$ to denote the moduli stack of de Rham $\check{G}$ -local systems on C . Similarly, for each finite-dimensional representation V of $\check{G}$, we denote by $V_{\mathrm{univ}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$ the universal local system associated to V , which lies in cohomological degree -2 .

By default, we work with stacks over $\mathbb{C}$ for the étale topology. For such a stack $\mathcal{Y}$, we denote by $\mathcal{Y}_{\mathrm{dR}}$ its de Rham stack. We denote by $\mathrm{D-mod}(\mathcal{Y}) := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}}) \cong \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}})$ its category of D -modules. We have the adjoint pairs

$$\mathrm{ind}^l : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \mathrm{D-mod}(\mathcal{Y}) : \mathrm{oblv}^l$$

and

$$\mathrm{ind}^r : \mathrm{IndCoh}(\mathcal{Y}) \rightleftarrows \mathrm{D-mod}(\mathcal{Y}) : \mathrm{oblv}^r$$

such that $\mathrm{ind}^l \cong \mathrm{ind}^r \circ (- \otimes \omega_{\mathcal{Y}})$ where $\omega_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ is the dualizing complex. We use $\mathrm{D}_{\mathcal{Y}} := \mathrm{ind}^l(\mathcal{O}_{\mathcal{Y}}) \cong \mathrm{ind}^r(\omega_{\mathcal{Y}}) \in \mathrm{D-mod}(\mathcal{Y})$ to denote the sheaf of differential operators on $\mathcal{Y}$.

Given a line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$, we also have the category of twisted D -modules $\mathrm{D-mod}_{\mathcal{L}}(\mathcal{Y})$ which is canonically equivalent to $\mathrm{D-mod}(\mathcal{Y})$. We have similar adjoint pairs

$$\mathrm{ind}_{\mathcal{L}}^l : \mathrm{QCoh}(\mathcal{Y}) \rightleftarrows \mathrm{D-mod}_{\mathcal{L}}(\mathcal{Y}) : \mathrm{oblv}_{\mathcal{L}}^l$$

and

$$\mathrm{ind}_{\mathcal{L}}^r : \mathrm{IndCoh}(\mathcal{Y}) \rightleftarrows \mathrm{D-mod}_{\mathcal{L}}(\mathcal{Y}) : \mathrm{oblv}_{\mathcal{L}}^r.$$

We use $\mathrm{D}_{\mathcal{Y},\mathcal{L}} := \mathrm{ind}_{\mathcal{L}}^l(\mathcal{O}_{\mathcal{Y}}) \in \mathrm{D-mod}_{\mathcal{L}}(\mathcal{Y})$. Under the identification $\mathrm{D-mod}_{\mathcal{L}}(\mathcal{Y}) \cong \mathrm{D-mod}(\mathcal{Y})$, we have $\mathrm{ind}_{\mathcal{L}}^l \cong \mathrm{ind}^l \circ (- \otimes \mathcal{L}^{-1})$ and other similar identifications.

We will fix a square root $\Omega_C^{1/2}$ of the canonical line bundle Ω_C on C , which gives a square root $\mathcal{L}_{G,\mathfrak{g}}^{1/2}$ of the line bundle $\mathcal{L}_{G,\mathfrak{g}}$ on Bun_G . We use κ to denote the critical level of G , and we use

$$\mathrm{D-mod}_{\kappa}(\mathrm{Bun}_G) := \mathrm{D-mod}_{\mathcal{L}_{G,\mathfrak{g}}^{1/2}}(\mathrm{Bun}_G)$$

to denote the category of twisted D -modules on Bun_G at the critical level, and we will sometimes write κ instead of $\mathcal{L}_{G,\mathfrak{g}}^{1/2}$ in our notation.

Given a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of stacks, we write $f_*, f^*, f^!$ to denote the usual functors on QCoh and IndCoh , and we write $f_{\mathrm{dR},*}, f_{\mathrm{dR}}^!$ to denote the usual functors on $\mathrm{D-mod}$, whenever they are defined. When $\mathcal{Y}$ is a point, we write $\Gamma(\mathcal{X}, -) := f_*(-)$ and $\Gamma_{\mathrm{dR}}(\mathcal{X}, -) := f_{\mathrm{dR},*}(-)$.

Acknowledgements. The authors are grateful to their advisor Zhiwei Yun for suggesting the problem and for constant encouragement and support.

AI usage. The authors originally had the idea of using similar methods to compute the Euler characteristics of the stalks of $\mathbb{L}_G(\mathcal{P}_X)$. Wang then prompted GPT-6Pro with this idea and asked whether stronger statements beyond Euler characteristics could be established. This led to Proposition 1.6. We also used GPT-6 and Claude Opus 5.5 for literature searches, most notably for Proposition 3.7, and for spotting typos and polishing the grammar. The article was written by humans, and the authors take full responsibility for any errors.

2. LOCAL CONSTANCY

2.1. Main result. In this section, we prove the following result:

Proposition 2.1 (Proposition 1.6). *The sheaf $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_G^{\mathrm{irr}}} \in \mathrm{QCoh}(\mathrm{Loc}_G^{\mathrm{irr}})$ has bounded cohomological degrees with locally free cohomologies of finite rank.*

We will deduce Proposition 2.1 from the following two lemmas:

Lemma 2.2. *The object $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_G^{\mathrm{irr}}}$ is a perfect complex.*

Lemma 2.3. *The object $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_G^{\mathrm{irr}}}$ admits (not necessarily flat) connections locally on $\mathrm{Loc}_G^{\mathrm{irr}}$.*

It is easy to see that Lemma 2.2 and Lemma 2.3 imply Proposition 2.1. Lemma 2.3 follows immediately from Proposition 2.6, which we prove below. We first prove Lemma 2.2.

Proof of Lemma 2.2. Let

$$j_{\mathrm{cusp}} : \mathrm{Bun}_G^{\mathrm{cusp}} \rightarrow \mathrm{Bun}_G \quad (2.1)$$

be the open immersion of the open substack where the Harder–Narasimhan filtration has slope $\lambda \in X_*(T)_{\mathbb{Q}}^+$ with $\lambda \leq (2g - 2)\rho$ where $\rho \in X_*(T)_{\mathbb{Q}}^+$ is the half-sum of positive coroots as defined in [DG16, §B.1.2]. Here the order is defined such that $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1$ is a linear combination of simple coroots of G with non-negative coefficients. By [DG16, Proposition 1.4.6], any object of $\mathrm{D-mod}(\mathrm{Bun}_G)_{\mathrm{cusp}}$ is a clean extension of its restriction to $\mathrm{Bun}_G^{\mathrm{cusp}}$. Denote $\mathcal{F} := j_{\mathrm{cusp}}^* \mathcal{P}_X \in \mathrm{D-mod}(\mathrm{Bun}_G^{\mathrm{cusp}})$. We have $\mathbb{L}_G(\mathcal{P}_X)|_{\mathrm{Loc}_G^{\mathrm{irr}}} \cong \mathbb{L}_G^{\mathrm{coarse}}((j_{\mathrm{cusp}})_! \mathcal{F})|_{\mathrm{Loc}_G^{\mathrm{irr}}}$.

First, we show that $\mathbb{L}_G^{\mathrm{coarse}}((j_{\mathrm{cusp}})_! \mathcal{F})$ has bounded cohomological degrees. Indeed, by [GR24a, Proposition 1.8.2, Theorem 2.1.2], we know that $\mathbb{L}_G^{\mathrm{coarse}}$ has bounded cohomological amplitude. Therefore, it suffices to show that $(j_{\mathrm{cusp}})_! \mathcal{F}$ has bounded cohomological degrees. This follows from the fact that $\mathcal{F} \in \mathrm{D-mod}(\mathrm{Bun}_G^{\mathrm{cusp}})$ has bounded cohomological degrees and that $(j_{\mathrm{cusp}})_!$ has bounded cohomological amplitude, where the latter follows from the argument in [GR24a, §2.2.4]: it is clear that $(j_{\mathrm{cusp}})_!$ is right t -exact, and the left cohomological amplitude is bounded by the minimal number of affine charts covering a smooth cover of $\mathrm{Bun}_G^{\mathrm{cusp}}$.

Assume $\mathcal{F} \in \mathrm{D-mod}(\mathrm{Bun}_G^{\mathrm{cusp}})^{\leq N}$ for some $N \in \mathbb{Z}$. Write $\mathrm{Bun}_G^{\mathrm{cusp}} = [Z/K]$ where Z is a smooth scheme and K is a smooth affine algebraic group, and let f denote the map $f : Z \rightarrow \mathrm{Bun}_G^{\mathrm{cusp}}$. We have $\mathcal{F} \cong \mathrm{colim}(\cdots f_! f^! f_! f^! \mathcal{F} \rightrightarrows f_! f^! \mathcal{F})$. Let $\mathcal{F}_d$ denote the d -skeleton of the simplicial object. Then $\mathrm{Cone}(\mathcal{F}_d \rightarrow \mathcal{F}) \in \mathrm{D-mod}(\mathrm{Bun}_G^{\mathrm{cusp}})^{\leq N-d}$. Since all $\mathcal{F}_d$ are compact, we know that $\mathbb{L}_G^{\mathrm{coarse}}((j_{\mathrm{cusp}})_! \mathcal{F}_d)|_{\mathrm{Loc}_G^{\mathrm{irr}}}$ are perfect complexes. Combining this with the cohomological boundedness of $\mathbb{L}_G^{\mathrm{coarse}} \circ (j_{\mathrm{cusp}})_!$, we obtain that $\mathbb{L}_G^{\mathrm{coarse}}((j_{\mathrm{cusp}})_! \mathcal{F})|_{\mathrm{Loc}_G^{\mathrm{irr}}}$ has coherent cohomologies and hence is a perfect complex. $\square$

2.2. Atiyah class.

2.2.1. First infinitesimal neighborhood. We first recall the construction of the first infinitesimal neighborhood of a morphism in [GR20, Chapter 9, §5.1]. Given a morphism between prestacks

$$f : \mathcal{Y} \rightarrow \mathcal{Z}$$

where $\mathcal{Y}$ and $\mathcal{Z}^\wedge_{\mathcal{Y}} := \mathcal{Z} \times_{\mathcal{Z}_{\mathrm{dR}}} \mathcal{Y}_{\mathrm{dR}}$ are laft and admit deformation theory, consider the associated map on tangent complexes¹

$$T_{\mathcal{Y}/\mathcal{Z}} \rightarrow T_{\mathcal{Y}}.$$

¹ $T_{\mathcal{Y}} \in \mathrm{IndCoh}(\mathcal{Y})$ should be regarded as the *Serre* dual of the cotangent complex $\mathbb{L}_{\mathcal{Y}}$, rather than the linear dual.

Using the construction in [GR20, Chapter 8, §5.1] this induces a square-zero extension²

$$f^{(1)} : \mathcal{Y} \rightarrow \mathcal{Y}_Z^{(1)}$$

equipped with a map

$$\mathcal{Y}_Z^{(1)} \rightarrow \mathcal{Z}.$$

This is the first infinitesimal neighborhood of the morphism f . Moreover, [GR20, Chapter 8, Proposition 6.4.2] gives a fiber sequence in $\mathrm{IndCoh}(\mathcal{Y}_Z^{(1)})$:³

$$(f^{(1)})_* T_{\mathcal{Y}/\mathcal{Z}} \rightarrow (f^{(1)})_* \omega_{\mathcal{Y}} \rightarrow \omega_{\mathcal{Y}_Z^{(1)}}. \quad (2.2)$$

This construction is functorial in f : Given a sequence of morphisms

$$\mathcal{Y} \xrightarrow{f} \mathcal{Z}_1 \xrightarrow{g} \mathcal{Z}_2,$$

there is an induced map

$$i_g : \mathcal{Y}_{\mathcal{Z}_1}^{(1)} \rightarrow \mathcal{Y}_{\mathcal{Z}_2}^{(1)}$$

and a commutative diagram in $\mathrm{IndCoh}(\mathcal{Y}_{\mathcal{Z}_2}^{(1)})$:

$$\begin{array}{ccccc} (i_g \circ f_{\mathcal{Z}_1}^{(1)})_* T_{\mathcal{Y}/\mathcal{Z}_1} & \longrightarrow & (i_g \circ f_{\mathcal{Z}_1}^{(1)})_* \omega_{\mathcal{Y}} & \longrightarrow & (i_g)_* \omega_{\mathcal{Y}_{\mathcal{Z}_1}^{(1)}} \\ \downarrow & & \downarrow & & \downarrow \\ (f_{\mathcal{Z}_2}^{(1)})_* T_{\mathcal{Y}/\mathcal{Z}_2} & \longrightarrow & (f_{\mathcal{Z}_2}^{(1)})_* \omega_{\mathcal{Y}} & \longrightarrow & \omega_{\mathcal{Y}_{\mathcal{Z}_2}^{(1)}} \end{array}. \quad (2.3)$$

The construction also commutes with base change by [GR20, Chapter 8, Proposition 5.4.3]: Given a Cartesian diagram of left prestacks admitting deformation theory

$$\begin{array}{ccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}} & \tilde{\mathcal{Z}} \\ \downarrow g_{\mathcal{Y}} & & \downarrow g_{\mathcal{Z}} \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{Z} \end{array},$$

one has a commutative diagram with Cartesian squares:

$$\begin{array}{ccccc} \tilde{\mathcal{Y}} & \xrightarrow{\tilde{f}^{(1)}} & \tilde{\mathcal{Y}}_{\tilde{\mathcal{Z}}}^{(1)} & \longrightarrow & \tilde{\mathcal{Z}} \\ \downarrow g_{\mathcal{Y}} & & \downarrow g_{\mathcal{Y}}^{(1)} & & \downarrow g_{\mathcal{Z}} \\ \mathcal{Y} & \xrightarrow{f^{(1)}} & \mathcal{Y}_{\mathcal{Z}}^{(1)} & \longrightarrow & \mathcal{Z} \end{array}$$

and a commutative diagram in $\mathrm{IndCoh}(\tilde{\mathcal{Y}}_{\tilde{\mathcal{Z}}}^{(1)})$:

$$\begin{array}{ccccc} (\tilde{f}^{(1)})_* T_{\tilde{\mathcal{Y}}/\tilde{\mathcal{Z}}} & \longrightarrow & (\tilde{f}^{(1)})_* \omega_{\tilde{\mathcal{Y}}} & \longrightarrow & \omega_{\tilde{\mathcal{Y}}_{\tilde{\mathcal{Z}}}^{(1)}} \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ (g_{\mathcal{Y}}^{(1)})^! (f^{(1)})_* T_{\mathcal{Y}/\mathcal{Z}} & \longrightarrow & (g_{\mathcal{Y}}^{(1)})^! (f^{(1)})_* \omega_{\mathcal{Y}} & \longrightarrow & (g_{\mathcal{Y}}^{(1)})^! \omega_{\mathcal{Y}_{\mathcal{Z}}^{(1)}} \end{array}. \quad (2.4)$$

²We will sometimes write $\mathcal{Y}^{(1)}$ rather than $\mathcal{Y}_Z^{(1)}$ when $\mathcal{Z}$ is clear from the context.

³The functor $f_*^{(1)}$ is defined since $f^{(1)}$ is inf-schematic, and we use the IndCoh -sheaf theory in [GR20, Chapter 3, Theorem 5.4.3]. The same applies to the other $*$ -pushforward functors in this section.

2.2.2. *General definition.* Applying the construction of §2.2.1 to the diagonal map

$$\Delta_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y},$$

one obtains the first infinitesimal neighborhood of the diagonal:

$$\mathcal{Y} \xrightarrow{\Delta_{\mathcal{Y}}^{(1)}} \mathcal{Y}^{(1)} \xrightarrow{(p_1^{(1)}, p_2^{(1)})} \mathcal{Y} \times \mathcal{Y}$$

and a fiber sequence in $\text{IndCoh}(\mathcal{Y}^{(1)})$:

$$(\Delta_{\mathcal{Y}}^{(1)})_* T_{\mathcal{Y}}[-1] \rightarrow (\Delta_{\mathcal{Y}}^{(1)})_* \omega_{\mathcal{Y}} \rightarrow \omega_{\mathcal{Y}^{(1)}}. \quad (2.5)$$

Regarding these as kernels gives a natural transformation between endofunctors on $\text{IndCoh}(\mathcal{Y})$:

$$\text{At}_{\mathcal{Y}} : T_{\mathcal{Y}}[-1] \otimes^! - \rightarrow \text{id}. \quad (2.6)$$

We call (2.6) (or (2.5)) the *Atiyah class* of $\mathcal{Y}$. For $\mathcal{F} \in \text{IndCoh}(\mathcal{Y})$, we denote by

$$\text{At}_{\mathcal{Y}}(\mathcal{F}) : T_{\mathcal{Y}}[-1] \otimes^! \mathcal{F} \rightarrow \mathcal{F}$$

the induced morphism.

For $\mathcal{F} \in \text{IndCoh}(\mathcal{Y})$, a *connection* on $\mathcal{F}$ is a null-homotopy of $\text{At}_{\mathcal{Y}}(\mathcal{F})$. To see that this is compatible with the usual definition of a connection, one notes that (2.5) gives a fiber sequence in $\text{IndCoh}(\mathcal{Y})$:

$$T_{\mathcal{Y}}[-1] \otimes^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow (p_2^{(1)})_*(p_1^{(1)})^! \mathcal{F}.$$

Therefore, a null-homotopy of $\text{At}_{\mathcal{Y}}(\mathcal{F})$ is equivalent to a splitting of the last map, which by adjunction is equivalent to a map $(p_1^{(1)})^! \mathcal{F} \rightarrow (p_2^{(1)})^! \mathcal{F}$ in $\text{IndCoh}(\mathcal{Y}^{(1)})$ whose restriction to $\mathcal{Y}$ is the identity. This is exactly the usual definition of a connection.

2.2.3. *Atiyah class of $\text{Loc}_{\check{G}}$.* Applying the construction in §2.2.2 to $\mathcal{Y} = \text{Loc}_{\check{G}}$, we obtain the Atiyah class of $\text{Loc}_{\check{G}}$ as a natural transformation between endofunctors on $\text{IndCoh}(\text{Loc}_{\check{G}})$:⁴

$$\text{At} : \Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}})[-2] \otimes - \cong T_{\text{Loc}_{\check{G}}}[-1] \otimes^! - \rightarrow \text{id}, \quad (2.7)$$

where we regard $\Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}}) \in \text{QCoh}(\text{Loc}_{\check{G}})$. It satisfies $\Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}})[-2] \otimes \omega_{\text{Loc}_{\check{G}}} \cong T_{\text{Loc}_{\check{G}}}[-1]$.

We also need the variant where one takes $\mathcal{Y} = \text{Loc}_{\check{G}} \times C_{\text{dR}}$ which gives an Atiyah class as a natural transformation between endofunctors on $\text{IndCoh}(\text{Loc}_{\check{G}} \times C_{\text{dR}})$:

$$\text{At}_C : (\Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}})[-2] \boxtimes \omega_{C_{\text{dR}}}) \otimes - \rightarrow \text{id}, \quad (2.8)$$

where we are using the fact that $T_{\text{Loc}_{\check{G}} \times C_{\text{dR}}} \cong T_{\text{Loc}_{\check{G}}} \boxtimes \omega_{C_{\text{dR}}}$.

2.3. Local Atiyah class.

2.3.1. *Spherical Hecke category.* Define the spherical Hecke category

$$\text{Sat}_G := \text{D-mod}(L^+G \backslash LG / L^+G)^{\text{ren}} \cong \text{IndCoh}(\mathbb{B}\check{G} \times_{[\check{\mathfrak{g}}/\check{G}]} \mathbb{B}\check{G}).$$

This can be made in families over C and defines a monoidal category

$$\text{Sat}_{G,C} := \text{D-mod}((L^+G)_C \backslash (LG)_C / (L^+G)_C)^{\text{ren}} \cong \text{IndCoh}(\text{Loc}_{\check{G},C}^{\text{reg}} \times_{\text{Loc}_{\check{G},C}^{\text{mer}}} \text{Loc}_{\check{G},C}^{\text{reg}})$$

where $\text{Loc}_{\check{G},C}^{\text{reg}}$ (and $\text{Loc}_{\check{G},C}^{\text{mer}}$) are informally defined as the moduli stacks classifying a point $c \in C$ together with a $\check{G}$ -local system on the formal disk (and punctured disk) around c . We refer to [ABC⁺24a, §B.7] for a rigorous definition.⁵ We use $\text{Sat} : \text{Rep}(\check{G}) \rightarrow \text{Sat}_G$ and $\text{Sat}_C : \text{Rep}(\check{G}) \rightarrow \text{Sat}_{G,C}$ to denote the obvious functors.

Moreover, one also considers the spherical Hecke category with loop rotation

$$\text{Sat}_{G,\hbar} := \text{D-mod}((L^+G \rtimes \text{Aut}(D)) \backslash (LG \rtimes \text{Aut}(D)) / (L^+G \rtimes \text{Aut}(D)))^{\text{ren}} \cong \text{Mod}_{U_{\hbar}(\check{\mathfrak{g}})}^{\check{G}} \quad (2.9)$$

where $\text{Aut}(D)$ is the group scheme of automorphisms of the disk $D = \text{Spec}(\mathbb{C}[[t]])$ fixing the origin. This is [BF08, Theorem 5]. We use $\text{Sat}_{\hbar} : \text{Rep}(\check{G}) \rightarrow \text{Sat}_{G,\hbar}$ to denote the natural functor.

⁴Note that under the identification $\text{D-mod}(\mathcal{Y}) \cong \text{QCoh}(\mathcal{Y}_{\text{dR}})$, the usual tensor product $\otimes$ on $\text{QCoh}(\mathcal{Y}_{\text{dR}})$ corresponds to the $!$ -tensor product on $\text{D-mod}(\mathcal{Y})$, and we by default use this tensor product.

⁵In loc. cit., these stacks are defined over Ran , while our $\text{Loc}_{\check{G},C}^{\text{reg}}$ and $\text{Loc}_{\check{G},C}^{\text{mer}}$ are their restrictions to the diagonal $C \subset \text{Ran}$.

We use $*$ to denote the monoidal operation given by convolution on Sat_G , $\text{Sat}_{G,C}$ and $\text{Sat}_{G,\hbar}$, and we use the same notation $*$ to denote the action of $\text{Sat}_{G,C}$ on $\text{D-mod}(\text{Bun}_G)$, $\text{D-mod}(\text{Bun}_G \times C)$ (as well as the equivalent action on the spectral side) understood in the usual way. Note that the compatibility of the actions under the equivalence $\mathbb{L}_G$ is a special case of [ABC⁺24a, Proposition 18.3.8] which proves the compatibility over the entire Ran space.

2.3.2. *Local Atiyah classes.* Koszul duality gives an equivalence of categories

$$\text{Sat}_G \cong \text{QCoh}([\check{\mathfrak{g}}^*/\check{G}])^\vee$$

where the shearing $//$ places $\check{\mathfrak{g}}^*$ in cohomological degree -2 . In particular, this description gives a tautological morphism

$$\text{At}^1 : \text{Sat}(\check{\mathfrak{g}}) \rightarrow \text{Sat}(\mathbb{C})[2]. \quad (2.10)$$

We call this the *local Atiyah class*.

We have a unique lift of the local Atiyah class to a morphism

$$\text{At}_\hbar^1 : \text{Sat}_\hbar(\check{\mathfrak{g}}) \rightarrow \text{Sat}_\hbar(\mathbb{C})[2] \quad (2.11)$$

also given tautologically by (2.9).

The following lemma gives a unique lift of the local Atiyah class to a family over the curve:

Lemma 2.4. *There is a contractible space of choices of a morphism*

$$\text{At}_C^1 : \text{Sat}_C(\check{\mathfrak{g}}) \rightarrow \text{Sat}_C(\mathbb{C})[2] \quad (2.12)$$

whose fiber at any point $c \in C$ is the local Atiyah class At^1 .

Proof. The existence follows from pulling back (2.11) on the automorphic side. Again on the automorphic side, one has

$$\text{Hom}^*(\text{Sat}_C(\check{\mathfrak{g}}), \text{Sat}_C(\mathbb{C})[2]) \cong \text{Hom}^*(\text{Sat}_\hbar(\check{\mathfrak{g}}), \text{Sat}_\hbar(\mathbb{C})[2]) \otimes_{\mathbb{C}[\hbar]} H_{\text{dR}}^*(C, \mathbb{C}).$$

Since $\text{Hom}^i(\text{Sat}_\hbar(\check{\mathfrak{g}}), \text{Sat}_\hbar(\mathbb{C})[2]) = 0$ for $i < 0$, we know that

$$\text{Hom}^0(\text{Sat}_C(\check{\mathfrak{g}}), \text{Sat}_C(\mathbb{C})[2]) \cong \text{Hom}^0(\text{Sat}_\hbar(\check{\mathfrak{g}}), \text{Sat}_\hbar(\mathbb{C})[2]) \cong \text{Hom}^0(\text{Sat}(\check{\mathfrak{g}}), \text{Sat}(\mathbb{C})[2]).$$

This implies the uniqueness of (2.12). $\square$

By abuse of notation, we also write At_C^1 for the induced natural transformation between endofunctors on $\text{IndCoh}(\text{Loc}_{\check{G}} \times C_{\text{dR}})$:

$$\text{At}_C^1 : \check{\mathfrak{g}}_{\text{univ}}[-2] \otimes - \rightarrow \text{id}. \quad (2.13)$$

2.3.3. *An alternative description of the local Atiyah class.* We first give an alternative description of the local Atiyah class (2.10). Applying the construction of §2.2.1 to the diagonal map

$$\Delta_{\mathbb{B}\check{G}} : \mathbb{B}\check{G} \rightarrow \mathbb{B}\check{G} \times_{[\check{\mathfrak{g}}/\check{G}]} \mathbb{B}\check{G},$$

we get a square-zero extension

$$\mathbb{B}\check{G} \xrightarrow{\Delta_{\mathbb{B}\check{G}}^{(1)}} (\mathbb{B}\check{G})^{(1)} \xrightarrow{(p_1^{(1)}, p_2^{(1)})} \mathbb{B}\check{G} \times_{[\check{\mathfrak{g}}/\check{G}]} \mathbb{B}\check{G}$$

and a fiber sequence in $\text{IndCoh}(\mathbb{B}\check{G}^{(1)})$:

$$(\Delta_{\mathbb{B}\check{G}}^{(1)})^*(T_{\mathbb{B}\check{G}/(\mathbb{B}\check{G} \times_{[\check{\mathfrak{g}}/\check{G}]} \mathbb{B}\check{G})}) \xrightarrow{\text{At}^1} (\Delta_{\mathbb{B}\check{G}}^{(1)})_* \omega_{\mathbb{B}\check{G}} \rightarrow \omega_{\mathbb{B}\check{G}^{(1)}}. \quad (2.14)$$

The first map is a lift of the local Atiyah class (2.10) in the sense that after applying $(p_1^{(1)}, p_2^{(1)})^*$ to (2.14) and using the identification

$$(\Delta_{\mathbb{B}\check{G}})_* T_{\mathbb{B}\check{G}/(\mathbb{B}\check{G} \times_{[\check{\mathfrak{g}}/\check{G}]} \mathbb{B}\check{G})} \cong \text{Sat}(\check{\mathfrak{g}})[-2],$$

it becomes (2.10).

We can also apply the construction in §2.2.1 to the diagonal map

$$\Delta_{\text{Loc}_{\check{G},C}^{\text{reg}}} : \text{Loc}_{\check{G},C}^{\text{reg}} \rightarrow \text{Loc}_{\check{G},C}^{\text{reg}} \times_{\text{Loc}_{\check{G},C}^{\text{mer}}} \text{Loc}_{\check{G},C}^{\text{reg}},$$

which gives a square-zero extension

$$\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \xrightarrow{\Delta_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}}^{(1)}} \mathrm{Loc}_{\check{G},C}^{\mathrm{reg},(1)} \rightarrow \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer}}} \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}$$

and a fiber sequence in $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G},C}^{\mathrm{reg},(1)})$:

$$(\Delta_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}}^{(1)})_*(T_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}/(\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer}}} \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}})}) \xrightarrow{\mathrm{At}_C^1} (\Delta_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}}^{(1)})_* \omega_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}} \rightarrow \omega_{(\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}})^{(1)}}. \quad (2.15)$$

where the first map is a lift of the Atiyah class in the same sense under the isomorphism

$$(\Delta_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}}^{(1)})_* T_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}/(\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer}}} \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}})} \cong \mathrm{Sat}_C(\check{\mathfrak{g}})[-2].$$

To see this isomorphism, we only need to show that there is an isomorphism in $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}) \cong \mathrm{IndCoh}(\mathbb{B}\check{G} \times C_{\mathrm{dR}})$:

$$T_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}/\mathrm{Loc}_{\check{G},C}^{\mathrm{mer}}} \cong \check{\mathfrak{g}}[-1] \boxtimes \omega_{C_{\mathrm{dR}}}.$$

This follows from [Roz21, Proposition 6.1.2] by pulling back along the map $\mathbb{B}\check{G} \rightarrow \mathrm{Loc}_{\check{G}}$ given by the trivial $\check{G}$ -local system on C .

2.4. Comparing local and global Atiyah classes. Now we compare the local Atiyah class with the global Atiyah class. The comparison is given by the following proposition:

Proposition 2.5. *One has the following commutative diagram of natural transformations between endofunctors on $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$:*

$$\begin{array}{ccc} \check{\mathfrak{g}}_{\mathrm{univ}}[-2] \otimes - & \xrightarrow{\mathrm{At}_C^1} & \mathrm{id} \\ \downarrow \mathrm{Gys} \otimes - & & \downarrow \sim \\ (\Gamma_{\mathrm{dR}}(C, \check{\mathfrak{g}}_{\mathrm{univ}})[-2] \boxtimes \omega_{C_{\mathrm{dR}}}) \otimes - & \xrightarrow{\mathrm{At}_C} & \mathrm{id} \end{array}$$

where $\mathrm{Gys} : \check{\mathfrak{g}}_{\mathrm{univ}}[-2] \rightarrow \Gamma_{\mathrm{dR}}(C, \check{\mathfrak{g}}_{\mathrm{univ}})[-2] \boxtimes \omega_{C_{\mathrm{dR}}} \in \mathrm{QCoh}(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$ is the Gysin map.

Proof. Since all the endofunctors are linear over $\mathrm{D}\text{-mod}(C)$, it suffices to compare the kernels of the natural transformations as objects in $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}} \times \mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$.

We first describe the local Atiyah class At_C^1 in terms of kernels. Let $\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}$ be the moduli stack of a point $c \in C$ and a $\check{G}$ -local system on $C \setminus c$. Using [ABC⁺24a, Lemma E.7.2], we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}} & \xrightarrow{\Delta_{\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}}^1} & (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \\ \downarrow & & \downarrow \\ \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} & \xrightarrow{\Delta_{\mathrm{Loc}_{\check{G},C}^{\mathrm{reg}}}^1} & \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer}}} \mathrm{Loc}_{\check{G},C}^{\mathrm{reg}} \end{array}. \quad (2.16)$$

Applying the construction of §2.2.1 to the upper horizontal map, we obtain a square-zero extension

$$\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}} \xrightarrow{(\Delta_{\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}}^1)^{(1)}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})^{(1)}_{(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})} \rightarrow (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$$

and a fiber sequence in $\mathrm{IndCoh}((\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})^{(1)})$:

$$(\Delta_{\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}}^1)^{(1)}_* T_{(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})/(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})} \xrightarrow{\mathrm{At}_C^1} (\Delta_{\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}}^1)^{(1)}_* \omega_{\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}} \rightarrow \omega_{(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})^{(1)}_{(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}}) \times_{\mathrm{Loc}_{\check{G},C}^{\mathrm{mer},g}} (\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})}} \quad (2.17)$$

in which the first map induces the local Atiyah class At_C^1 at the level of kernels by unwinding the definition and using the base change property of the first infinitesimal neighborhood (2.4) applied to (2.16).

Now we compare the local Atiyah class At_C^1 with the global Atiyah class At_C . Consider the sequence of morphisms

$$\text{Loc}_{\check{G}} \times C_{\text{dR}} \xrightarrow{\Delta_{\text{Loc}_{\check{G}} \times C_{\text{dR}}}^1} (\text{Loc}_{\check{G}} \times C_{\text{dR}}) \times_{\text{Loc}_{\check{G}, C}^{\text{mer}, g}} (\text{Loc}_{\check{G}} \times C_{\text{dR}}) \xrightarrow{\Delta_{\text{Loc}_{\check{G}} \times C_{\text{dR}}}^{1 \rightarrow g}} \text{Loc}_{\check{G}} \times \text{Loc}_{\check{G}} \times C_{\text{dR}}$$

and apply the functoriality of the first infinitesimal neighborhood. We obtain an induced map

$$i_{\Delta_{\text{Loc}_{\check{G}} \times C_{\text{dR}}}^{1 \rightarrow g}} : (\text{Loc}_{\check{G}} \times C_{\text{dR}})_{(\text{Loc}_{\check{G}} \times C_{\text{dR}}) \times_{\text{Loc}_{\check{G}, C}^{\text{mer}, g}} (\text{Loc}_{\check{G}} \times C_{\text{dR}})}^{(1)} \rightarrow (\text{Loc}_{\check{G}} \times C_{\text{dR}})_{\text{Loc}_{\check{G}} \times \text{Loc}_{\check{G}} \times C_{\text{dR}}}^{(1)}$$

and also a commutative diagram (2.3). This reduces the problem to showing the following commutative diagram in $\text{IndCoh}(\text{Loc}_{\check{G}} \times C_{\text{dR}})$:

$$\begin{array}{ccc} T_{(\text{Loc}_{\check{G}} \times C_{\text{dR}})/(\text{Loc}_{\check{G}} \times C_{\text{dR}}) \times_{\text{Loc}_{\check{G}, C}^{\text{mer}, g}} (\text{Loc}_{\check{G}} \times C_{\text{dR}})} & \xrightarrow{\sim} & \check{\mathfrak{g}}_{\text{univ}}[-2] \otimes \omega_{\text{Loc}_{\check{G}} \times C_{\text{dR}}} \\ \downarrow & & \downarrow \text{Gys} \otimes \text{id} \\ T_{(\text{Loc}_{\check{G}} \times C_{\text{dR}})/(\text{Loc}_{\check{G}} \times \text{Loc}_{\check{G}} \times C_{\text{dR}})} & \xrightarrow{\sim} & (\Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}})[-2] \boxtimes \omega_{C_{\text{dR}}}) \otimes \omega_{\text{Loc}_{\check{G}} \times C_{\text{dR}}} \end{array} .$$

This follows from [Roz21, Proposition 5.3.3(b)] by a shift. $\square$

2.5. L -observables. We will prove the following result in this section:

Proposition 2.6. *The morphism $\text{At}(\mathbb{L}_G(\mathcal{P}_X)) : \Gamma_{\text{dR}}(C, \check{\mathfrak{g}}_{\text{univ}})[-2] \otimes \mathbb{L}_G(\mathcal{P}_X) \rightarrow \mathbb{L}_G(\mathcal{P}_X)$ is null-homotopic locally on $\text{Loc}_{\check{G}}^{\text{irr}}$.*

This immediately implies Lemma 2.3. The proof of Proposition 2.6 will be given in §2.5.4.

2.5.1. Plancherel algebra. Consider the relative derived Satake category

$$\text{Sat}_X^{\text{nonren}} := \text{Shv}(LX/L^+G)$$

where $\text{Shv}(LX/L^+G)$ is the category of ind-constructible sheaves on LX/L^+G as defined in [BZSV24, §B.7].⁶ Let $\text{Sat}_G^{\text{nonren}} := \text{Shv}(L^+G \backslash LG/L^+G) \subset \text{Sat}_G$ be the non-renormalized spherical Hecke category. Then $\text{Sat}_X^{\text{nonren}}$ is a module category over $\text{Sat}_G^{\text{nonren}}$. It has a loop rotation version

$$\text{Sat}_{X, \hbar}^{\text{nonren}} := \text{Shv}(LX/(L^+G \rtimes \text{Aut}(D)))$$

which is a module category over $\text{Sat}_{G, \hbar}^{\text{nonren}}$.

Consider the map $i_X : L^+X/L^+G \rightarrow LX/L^+G$ and let $\delta_X := (i_X)_* \omega_{L^+X/L^+G} \in \text{Sat}_X^{\text{nonren}}$, and similarly define $\delta_{X, \hbar} \in \text{Sat}_{X, \hbar}^{\text{nonren}}$. The Plancherel algebra is defined as

$$\text{PL}_X := \underline{\text{End}}_{\text{Rep}(\check{G})}(\delta_X)$$

and it admits a quantization

$$\text{PL}_{X, \hbar} := \underline{\text{End}}_{\text{Rep}(\check{G})}(\delta_{X, \hbar}).$$

For our choice of X , [BFT22, Theorem 2.2.1, Theorem 3.1.1] gives isomorphisms

$$\text{PL}_X \cong \mathcal{O}(\check{M})^\flat \tag{2.18}$$

and

$$\text{PL}_{X, \hbar} \cong \mathcal{O}(\check{M})_{\hbar}^\flat, \tag{2.19}$$

where the shearing places $\check{M}$ in cohomological degree -1 and $\mathcal{O}(\check{M})_{\hbar}^\flat$ is the Weyl algebra of $\check{M}$ with shearing.

In particular, this gives a morphism

$$\mathfrak{c}_{\check{M}^*}^1 : \text{Sat}_{\hbar}(\check{M}^*) * \delta_{X, \hbar} \rightarrow \delta_{X, \hbar}[1] \tag{2.20}$$

Remark 2.7. (2.20) is a special case of the *local special cohomological correspondences* constructed in [LW25, §5.1]. While the description above determines $\mathfrak{c}_{\check{M}^*}^1$ only up to a scalar, the fundamental class of relative affine Grassmannian gives a canonical choice. Here we can choose $\mathfrak{c}_{\check{M}^*}^1$ such that Proposition 2.9 holds.

⁶This category is denoted $\text{SHV}_s^!(LX/L^+G)$ in [BZSV24, §B.7].

2.5.2. *The Clifford algebra action.* Now we apply the construction of [LW25] to obtain a Clifford algebra action on $\mathbb{L}_G(\mathcal{P}_X)$. Note that [LW25] works with a sheaf theory different from the one we are using in two aspects:

- (1) They work in the setting of ℓ -adic sheaves, while we work with D -modules. This is not a problem as all the constructions in [LW25] can be carried out in the setting of ind-holonomic D -modules.
- (2) They used a version of $*$ -sheaf theory (in which $*$ -pullback is universally defined) instead of $!$ -sheaf theory (in which $!$ -pullback is universally defined). However, both sheaf theories define the same Plancherel algebra, which only depends on finite type geometry (see [BZSV24, Remark 8.3.2]). In fact, the only input from [LW25] used in the article will be Proposition 2.8 and Corollary 2.10, which involve only algebraic stacks on the automorphic side.

By the local-to-global construction in [LW25, §4.4], one obtains a morphism

$$\mathbf{c}_{\check{M}^*}^g : \text{Sat}_C(\check{M}^*) * (\mathcal{P}_X \boxtimes \omega_{C_{\text{dR}}}) \rightarrow \mathcal{P}_X \boxtimes \omega_{C_{\text{dR}}}[1]. \quad (2.21)$$

Moreover, taking pushforward along C gives a morphism

$$\mathbf{c}_{\check{M}^*} : \text{Sat}_C(\check{M}^*) * \mathcal{P}_X \rightarrow \mathcal{P}_X[1]. \quad (2.22)$$

Translated to the spectral side, (2.22) becomes

$$\mathbf{c}_{\check{M}^*} : \Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)[-1] \otimes \mathbb{L}_G(\mathcal{P}_X) \rightarrow \mathbb{L}_G(\mathcal{P}_X). \quad (2.23)$$

Recall that $\check{M}$ is a symplectic representation of $\check{G}$. Therefore, the cohomology group $\Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)[-1]$ carries a symmetric bilinear form

$$\cup : \Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)^{\otimes 2}[-2] \rightarrow \mathbb{C}.$$

Let

$$\text{sw} : \Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)^{\otimes 2} \rightarrow \Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)^{\otimes 2}$$

be the swap map with the usual Koszul sign rule. The following result is a direct translation of [LW25, Corollary 5.7] to the spectral side:

Proposition 2.8. *One has*

$$\mathbf{c}_{\check{M}^*} \circ \mathbf{c}_{\check{M}^*} - \mathbf{c}_{\check{M}^*} \circ \mathbf{c}_{\check{M}^*} \circ (\text{sw} \otimes \text{id}) = c \cdot \cup \otimes \text{id} \in \text{Hom}^0(\Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)^{\otimes 2}[-2] \otimes \mathbb{L}_G(\mathcal{P}_X), \mathbb{L}_G(\mathcal{P}_X))$$

where $c \in \mathbb{C}^\times$ is a non-zero constant. In particular, the map $\mathbf{c}_{\check{M}^*}|_{\text{Loc}_G^\circ}$ induces an action of the Clifford algebra

$$\mathbb{O}_{\check{M}}^\circ := \text{Cl}(\Gamma_{\text{dR}}(C, \check{M}_{\text{univ}}^*)[-1]|_{\text{Loc}_G^\circ})$$

on $\mathbb{L}_G(\mathcal{P}_X)|_{\text{Loc}_G^\circ}$.

2.5.3. *The moment map identity.* Consider the moment map $\mu_{\check{M}} : \check{M} \rightarrow \check{\mathfrak{g}}^*$. It is a quadratic map, so it induces a morphism $\mu_{\check{M}}^* : \check{\mathfrak{g}} \rightarrow \text{Sym}^2 \check{M}^* \subset (\check{M}^*)^{\otimes 2}$. The following result can be directly deduced from [BFT22]:

Proposition 2.9. *We have the following commutative diagram in $\text{Sat}_{X, \hbar}^{\text{nonren}}$:*

$$\begin{array}{ccc} \text{Sat}_{\hbar}(\check{\mathfrak{g}}) * \delta_{X, \hbar} & \xrightarrow{\text{At}_{\hbar}^! \text{id}} & \text{Sat}_{\hbar}(\mathbb{C})[2] * \delta_{X, \hbar} \\ \downarrow \text{Sat}_{\hbar}(\mu_{\check{M}}^*)^* \text{id} & & \downarrow \sim \\ \text{Sat}_{\hbar}((\check{M}^*)^{\otimes 2}) * \delta_{X, \hbar} & \xrightarrow{\mathbf{c}_{\check{M}^*}^! \circ \mathbf{c}_{\check{M}^*}^!} & \delta_{X, \hbar}[2] \end{array}$$

where $\mathbf{c}_{\check{M}^*}^! \circ \mathbf{c}_{\check{M}^*}^! : \text{Sat}_{\hbar}((\check{M}^*)^{\otimes 2}) * \delta_{X, \hbar} \rightarrow \delta_{X, \hbar}[2]$ is the map

$$\text{Sat}_{\hbar}((\check{M}^*)^{\otimes 2}) * \delta_{X, \hbar} \cong \text{Sat}_{\hbar}(\check{M}^*) * \text{Sat}_{\hbar}(\check{M}^*) * \delta_{X, \hbar} \xrightarrow{\text{id} * \mathbf{c}_{\check{M}^*}^!} \text{Sat}_{\hbar}(\check{M}^*) * \delta_{X, \hbar}[1] \xrightarrow{\mathbf{c}_{\check{M}^*}^!} \delta_{X, \hbar}[2].$$

Proof. After reduction modulo $\hbar$, this follows from [BFT22, Theorem 2.2.1(a)]. This implies the quantized version since

$$\text{Hom}_{\text{Sat}_{X, \hbar}^{\text{nonren}}}^0(\text{Sat}_{\hbar}(\check{\mathfrak{g}}) * \delta_{X, \hbar}, \delta_{X, \hbar}) = 0$$

which can be seen from (2.19). □

Applying the local-to-global construction in [LW25, §4.4], we obtain the following global version of Proposition 2.9:

Corollary 2.10. *We have the following commutative diagram in $\mathrm{D}\text{-mod}(\mathrm{Bun}_G \times C)$:*

$$\begin{array}{ccc} \mathrm{Sat}_C(\check{\mathfrak{g}}) * (\mathcal{P}_X \boxtimes \omega_{C_{\mathrm{dR}}}) & \xrightarrow{\mathrm{At}_C^! \mathrm{id}} & \mathrm{Sat}_C(\mathbb{C})[2] * (\mathcal{P}_X \boxtimes \omega_{C_{\mathrm{dR}}}) \\ \downarrow \mathrm{Sat}_C(\mu_M^*) * \mathrm{id} & & \downarrow \sim \\ \mathrm{Sat}_C((\check{M}^*)^{\otimes 2}) * (\mathcal{P}_X \boxtimes \omega_{C_{\mathrm{dR}}}) & \xrightarrow{\mathfrak{c}_{\check{M}^*}^g \circ \mathfrak{c}_{\check{M}^*}^g} & \mathcal{P}_X \boxtimes \omega_{C_{\mathrm{dR}}}[2] \end{array}.$$

This can be further translated to the spectral side, which gives the following result:

Corollary 2.11. *We have the following commutative diagram in $\mathrm{IndCoh}(\mathrm{Loc}_{\check{G}} \times C_{\mathrm{dR}})$:*

$$\begin{array}{ccc} \check{\mathfrak{g}}_{\mathrm{univ}} \otimes (\mathbb{L}_G(\mathcal{P}_X) \boxtimes \omega_{C_{\mathrm{dR}}}) & \xrightarrow{\mathrm{At}_G^! (\mathbb{L}_G(\mathcal{P}_X) \boxtimes \omega_{C_{\mathrm{dR}}})} & \mathbb{L}_G(\mathcal{P}_X) \boxtimes \omega_{C_{\mathrm{dR}}}[2] \\ \downarrow (\mu_M^*)_{\mathrm{univ}} \otimes \mathrm{id} & & \downarrow \sim \\ (\check{M}_{\mathrm{univ}}^*)^{\otimes 2} \otimes (\mathbb{L}_G(\mathcal{P}_X) \boxtimes \omega_{C_{\mathrm{dR}}}) & \xrightarrow{\mathfrak{c}_{\check{M}^*}^g \circ \mathfrak{c}_{\check{M}^*}^g} & \mathbb{L}_G(\mathcal{P}_X) \boxtimes \omega_{C_{\mathrm{dR}}}[2] \end{array}.$$

Taking cohomology along C and applying Proposition 2.5, we obtain the following result:

Corollary 2.12. *We have the following commutative diagram in $\mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$:*

$$\begin{array}{ccc} \Gamma_{\mathrm{dR}}(C, \check{\mathfrak{g}}_{\mathrm{univ}})[-2] \otimes \mathbb{L}_G(\mathcal{P}_X) & \xrightarrow{\mathrm{At}(\mathbb{L}_G(\mathcal{P}_X))} & \mathbb{L}_G(\mathcal{P}_X) \\ \downarrow \Gamma_{\mathrm{dR}}(C, (\mu_M^*)_{\mathrm{univ}}) \otimes \mathrm{id} & & \downarrow \sim \\ \Gamma_{\mathrm{dR}}(C, (\check{M}_{\mathrm{univ}}^*)^{\otimes 2})[-2] \otimes \mathbb{L}_G(\mathcal{P}_X) & & \\ \downarrow (\Delta_C)_! \otimes \mathrm{id} & & \downarrow \\ \Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)^{\otimes 2}[-2] \otimes \mathbb{L}_G(\mathcal{P}_X) & \xrightarrow{\mathfrak{c}_{\check{M}^*}^g \circ \mathfrak{c}_{\check{M}^*}^g} & \mathbb{L}_G(\mathcal{P}_X) \end{array}$$

where $(\Delta_C)_! : \Gamma_{\mathrm{dR}}(C, (\check{M}_{\mathrm{univ}}^*)^{\otimes 2}) \rightarrow \Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)^{\otimes 2}$ is the Gysin homomorphism.

2.5.4. *Proof of Proposition 2.6.* By Corollary 2.12, it suffices to show that the morphism in $\mathrm{QCoh}(\mathrm{Loc}_{\check{G}})$

$$(\Delta_C)_! \circ \Gamma_{\mathrm{dR}}(C, (\mu_M^*)_{\mathrm{univ}}) : \Gamma_{\mathrm{dR}}(C, \check{\mathfrak{g}}_{\mathrm{univ}})[-2] \rightarrow \Gamma_{\mathrm{dR}}(C, \check{M}_{\mathrm{univ}}^*)^{\otimes 2}[-2]$$

is null-homotopic locally on $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}$. This is obvious as the source is concentrated in cohomological degree 1 while the target is concentrated in cohomological degree 0 on the classical stack $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}}$.

3. CLASSICAL LIMIT

In this section, we study the geometry of the relative Hitchin fibration, which serves as the microlocal picture of the period integral computation in §4.

3.1. The relative Hitchin fibration. Let $\mathrm{Higgs}_G := \mathrm{Map}(C, \mathfrak{g}_{\Omega_C}^*/G) \cong T^*\mathrm{Bun}_G$ be the Hitchin moduli stack, and let $\mathrm{Hitch}_G := \mathrm{Map}(C, \mathfrak{g}_{\Omega_C}^* // G)$ be the Hitchin base. Let $h_G : \mathrm{Higgs}_G \rightarrow \mathrm{Hitch}_G$ be the Hitchin fibration.

3.1.1. *Geometry of the relative Hitchin fibration.* The relative Dolbeault moduli space attached to the G -Hamiltonian space $M = T^*X$ is defined as

$$\mathrm{Higgs}_G^M = \mathrm{Map}(C, [M_{\Omega_C}/G]) \cong \mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}} := \mathrm{Map}(C, [\mathfrak{h}'_{\Omega_C}/H])$$

where the subscript Ω_C stands for twisting by the line bundle Ω_C via the obvious $\mathbb{G}_m$ -action as considered in Theorem 1.5, and $\mathfrak{h}' = \mathrm{Lie}(H')$ satisfies $\mathfrak{h}'|_H = \mathfrak{h} \oplus \mathrm{Std} \in \mathrm{Rep}(H)$ and $\mathbb{G}_m$ acts on $\mathfrak{h}'$ by dilations.

We have a commutative diagram

$$\begin{array}{ccc} M/G & \xrightarrow{\mu_M} & \mathfrak{g}^*/G \\ \downarrow & \swarrow & \\ M // G \cong \mathfrak{g}^* // G & & \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccc} \text{Higgs}_G^M & \longrightarrow & \text{Higgs}_G \\ \downarrow h_M & \swarrow h_G & \\ \text{Hitch}_G & & \end{array}.$$

Here the isomorphism $M // G \cong \mathfrak{g}^* // G$ follows from the fact that M is a *strongly tempered* G -Hamiltonian space in the sense that $\tilde{M}$ is a symplectic representation of $\tilde{G}$. In this case, one has the following result:

Proposition 3.1. *The map $h_M : \text{Higgs}_G^M \rightarrow \text{Hitch}_G$ is generically finite of degree $2^{(g-1)\dim(\tilde{M})}$.*

Proof. This follows from [HLM24, Corollary 6.1]. It can also be proved directly using the spectral curve description of Higgs_G^M . $\square$

3.1.2. Coherence of pushforward. Recall that we have fixed a square root $\Omega_C^{1/2} \in \text{Pic}(C)$ of the canonical line bundle, which gives us a canonical square root $\mathcal{L}_{G,\mathfrak{g}}^{1/2} \in \text{Pic}(\text{Bun}_G)$ of the determinant line bundle $\mathcal{L}_{G,\mathfrak{g}}$ on Bun_G for any reductive group G . Since $\mathfrak{h}'|_H = \mathfrak{h} \oplus \text{Std} \in \text{Rep}(H)$, we get a canonical square root $\mathcal{L}_{H,\text{Std}}^{1/2} := \mathcal{L}_{H,\mathfrak{h}'}^{1/2}|_{\text{Bun}_H} \otimes \mathcal{L}_{H,\mathfrak{h}}^{-1/2} \in \text{Pic}(\text{Bun}_H)$ of the line bundle $\mathcal{L}_{H,\text{Std}}$. Via the map $a : \text{Higgs}_G^M \cong \text{Bun}_H^{\mathfrak{h}'_{\Omega_C}} \rightarrow \text{Bun}_H$, we regard $\mathcal{L}_{H,\text{Std}}^{1/2}$ as also a line bundle on Higgs_G^M .

The main result of this section is the following theorem:

Theorem 3.2. *The object $(h_M)_*(\mathcal{L}_{H,\text{Std}}^{-1/2}) \in \text{QCoh}(\text{Hitch}_G)$ is a perfect complex.*

The idea of the proof of Theorem 3.2 is standard: using the Harder–Narasimhan stratification of Higgs_G^M (see §3.2), we reduce to the semistable locus. Its good moduli space is proper over Hitch_G , giving the required coherence. That is, consider the open substack $\text{Higgs}_G^{M,\text{ss}} := \text{Bun}_H^{\mathfrak{h}'_{\Omega_C},\text{ss}} \subset \text{Higgs}_G^M$ in §3.2. Let $h_M^{\text{ss}} : \text{Higgs}_G^{M,\text{ss}} \rightarrow \text{Hitch}_G$ be the restriction of h_M . We first prove the following:

Lemma 3.3. *The canonical map $(h_M)_*(\mathcal{L}_{H,\text{Std}}^{-1/2}) \rightarrow (h_M^{\text{ss}})_*(\mathcal{L}_{H,\text{Std}}^{-1/2}|_{\text{Higgs}_G^{M,\text{ss}}})$ is an isomorphism.*

Then, we can deduce the perfectness from the following which is a special case of Proposition 3.7, and the fact that Higgs_G^M is quasi-smooth:

Lemma 3.4. *The following is true:*

- (1) *The underlying classical stack $\text{Higgs}_G^{M,\text{ss},\text{cl}}$ of $\text{Higgs}_G^{M,\text{ss}}$ admits a good moduli space $\phi : \text{Higgs}_G^{M,\text{ss},\text{cl}} \rightarrow \text{Higgs}_G^{M,\text{ss},\text{cl},\text{coarse}}$ in the sense of [Alp13, Definition 4.1]. In particular, ϕ_* preserves coherent sheaves.*
- (2) *The induced map $h_M^{\text{ss},\text{cl},\text{coarse}} : \text{Higgs}_G^{M,\text{ss},\text{cl},\text{coarse}} \rightarrow \text{Hitch}_G$ is proper.*

3.1.3. A technical variant. For later application, we also need a technical variant of Lemma 3.3. Let

$$j_{\text{bdd}} : \text{Bun}_H^{\text{bdd}} \hookrightarrow \text{Bun}_H \tag{3.1}$$

be the open substack where the Harder–Narasimhan filtration has slope $\lambda \in X_*(T_H)_{\mathbb{Q}}^+$ with $\lambda \leq (6g-6)\rho_H$ where ρ_H is the half-sum of positive coroots of H and T_H is the maximal torus of H . Let $\text{Higgs}_G^{M,\text{bdd}}$ be its preimage under $\text{Bun}_H^{\mathfrak{h}'_{\Omega_C}} \rightarrow \text{Bun}_H$. Let $h_M^{\text{bdd}} : \text{Higgs}_G^{M,\text{bdd}} \rightarrow \text{Hitch}_G$ be the restriction of h_M . We have the following lemma:

Lemma 3.5. *The canonical map $(h_M)_*(\mathcal{L}_{H,\text{Std}}^{-1/2}) \rightarrow (h_M^{\text{bdd}})_*(\mathcal{L}_{H,\text{Std}}^{-1/2}|_{\text{Higgs}_G^{M,\text{bdd}}})$ is an isomorphism.*

For the application in §4, we need the following variant of Theorem 3.2, which is immediate from Theorem 3.2, Lemma 3.5, and Proposition 3.1:

Corollary 3.6. *The object $(h_M^{\text{bdd}})_*(\mathcal{L}_{H,\text{Std}}^{-1/2}|_{\text{Higgs}_G^{M,\text{bdd}}}) \in \text{QCoh}(\text{Hitch}_G)$ is a perfect complex, generically concentrated in cohomological degree zero and of rank $2^{(g-1)\dim(\tilde{M})}$.*

The proofs of Lemma 3.5 and Lemma 3.3 are in §3.3 and §3.4, respectively. They are direct applications of the method of [HLH23, Context 6.5, Proposition 6.6] using the Harder–Narasimhan stratification of Higgs_G^M (and Bun_H) established in §3.2.

3.2. Harder–Narasimhan stratification. Let H be a connected semisimple group over $\mathbb{C}$ ⁷ with a maximal torus T and a Borel subgroup $B \supset T$, and let V be a finite-dimensional representation of H . Let $\mathcal{L}$ be a line bundle on C . We want to equip $\text{Bun}_H^{V\mathcal{L}} := \text{Map}(C, [V_{\mathcal{L}}/H])$ with a stratification. Let $\text{Hitch}_H^{V\mathcal{L}} := \text{Map}(C, V_{\mathcal{L}} // H)$, and let $h_H^{V\mathcal{L}} : \text{Bun}_H^{V\mathcal{L}} \rightarrow \text{Hitch}_H^{V\mathcal{L}}$ be the natural map.

For a connected reductive group L and an L -bundle $\mathcal{E}_L$, the Tannakian description of L -bundles gives the degree $\deg(\mathcal{E}_L) \in X_*(Z(L))_{\mathbb{Q}}$.

Let $\lambda \in X_*(T)_{\mathbb{Q}}^+$. For $U \in \text{Rep}(H)$, we write $U^{\lambda \geq 0}$ for the sum of the weight spaces of non-negative weight with respect to λ , and we define $U^{\lambda > 0}$, $U^{\lambda = 0}$, $U^{\lambda \leq 0}$, $U^{\lambda < 0}$ similarly. Let P_{λ} be the parabolic subgroup of H with $\mathfrak{p}_{\lambda} := \text{Lie}(P_{\lambda}) = \mathfrak{h}^{\lambda \geq 0}$, and let L_{λ} be its Levi subgroup containing T . We have the following diagram:

$$\begin{array}{ccc} \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}} & \xrightarrow{p_{\lambda}} & \text{Bun}_H^{V\mathcal{L}} \\ s_{\lambda} \nearrow & & \downarrow q_{\lambda} \\ & & \text{Bun}_{L_{\lambda}}^{V^{\lambda=0}\mathcal{L}} \end{array}$$

For $y \in \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}}$, we put $\deg(y) := \deg(q_{\lambda}(y)) \in X_*(Z(L_{\lambda}))_{\mathbb{Q}} \subset X_*(T)_{\mathbb{Q}}$.

Let $(-, -)$ be the bilinear form on $X_*(T)_{\mathbb{Q}}$ induced from the Killing form. For $\lambda \in X_*(T)_{\mathbb{Q}}$, put $\|\lambda\| = (\lambda, \lambda)^{1/2}$.

Let $K/\mathbb{C}$ be an algebraically closed field. For $y \in \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}}(K)$, define

$$\mu(y) = \begin{cases} 0 & \text{if } \lambda = 0, \\ \frac{(\deg(y), \lambda)}{\|\lambda\|} & \text{otherwise.} \end{cases}$$

For $x \in \text{Bun}_H^{V\mathcal{L}}(K)$, define

$$M(x) = \sup\{\mu(y) \mid \lambda \in X_*(T)_{\mathbb{Q}}^+, y \in p_{\lambda}^{-1}(x)\},$$

where $p_{\lambda}^{-1}(x)$ denotes the set of $y \in \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}}(K)$ with $p_{\lambda}(y) \simeq x$. One has $M(x) \geq 0$ for every $x \in \text{Bun}_H^{V\mathcal{L}}(K)$. If $M(x) = 0$, we say that x is *semistable*.

We say that $y \in \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}}(K)$ is the *Harder–Narasimhan filtration* of $x = p_{\lambda}(y)$ if $M(x) = \mu(y) = \|\lambda\|$.

More generally, for a reductive group L (in our applications, a Levi subgroup of H) for $\theta \in X_*(Z(L))_{\mathbb{Q}}$, replacing $\deg(y)$ by $\deg(y) - \theta$ in the definition of $\mu(y)$ defines θ -semistability.

The following result combines results from [HLH23] and [Sch08].

Proposition 3.7.

- (1) For every $\lambda \in X_*(T)_{\mathbb{Q}}^+$, the λ -semistable points form an open substack $\text{Bun}_{L_{\lambda}}^{V^{\lambda=0}, \lambda\text{-ss}} \subset \text{Bun}_{L_{\lambda}}^{V^{\lambda=0}\mathcal{L}}$ with quasi-compact connected components. In particular, the semistable points form a quasi-compact open substack $\text{Bun}_H^{V\mathcal{L}, \text{ss}} \subset \text{Bun}_H^{V\mathcal{L}}$.
- (2) Define $\text{Bun}_H^{V\mathcal{L}, (\lambda)} := q_{\lambda}^{-1}(\text{Bun}_{L_{\lambda}}^{V^{\lambda=0}, \lambda\text{-ss}})$. The map $p_{\lambda} : \text{Bun}_{P_{\lambda}}^{V^{\lambda \geq 0}\mathcal{L}} \rightarrow \text{Bun}_H^{V\mathcal{L}}$ restricts to a locally closed immersion $i_{\lambda} : \text{Bun}_H^{V\mathcal{L}, (\lambda)} \rightarrow \text{Bun}_H^{V\mathcal{L}}$ on each connected component. These locally closed substacks form a stratification of $\text{Bun}_H^{V\mathcal{L}}$. More precisely, the numerical invariant μ defines a Θ -stratification in the sense of [HL22, Theorem A].
- (3) The underlying classical stack $\text{Bun}_H^{V\mathcal{L}, \text{ss}, \text{cl}}$ of $\text{Bun}_H^{V\mathcal{L}, \text{ss}}$ admits a good moduli space $\phi : \text{Bun}_H^{V\mathcal{L}, \text{ss}, \text{cl}} \rightarrow \text{Bun}_H^{V\mathcal{L}, \text{ss}, \text{cl}, \text{coarse}}$ in the sense of [Alp13, Definition 4.1]. In particular, ϕ_* preserves coherent sheaves.
- (4) The induced map $h_H^{V\mathcal{L}, \text{cl}, \text{coarse}} : \text{Bun}_H^{V\mathcal{L}, \text{ss}, \text{cl}, \text{coarse}} \rightarrow \text{Hitch}_H^{V\mathcal{L}}$ is proper.

Remark 3.8. Proposition 3.7 is an $\mathcal{L}$ -twisted version of the special case of [HLH23, Theorem 1.2] where the G -scheme X considered in *loc. cit.* is a linear representation.

Proof of Proposition 3.7. We only need to check that μ defines a Θ -stratification. Given this, (1) and (3) follow from [Sch08, Theorem 2.8.1.2] and the fact that affine GIT quotients are good moduli spaces (see, for

⁷Everything works for reductive groups over any field of characteristic zero with mild changes.

example, [Alp13, Remark 4.8]). (4) follows from [Sch08, Proposition 2.8.1.4]. The description of the strata using λ -semistable loci of the Levi subgroups follows from the same argument as in [HLH23, Proposition 5.10].

To prove that μ defines a Θ -stratification, we need to check the conditions in [HL22, Theorem A] and we will freely use the notations in loc. cit.

The condition $(\dagger\dagger)$ follows from that of Bun_H and the fact that $\text{Bun}_H^{V\mathcal{L}} \rightarrow \text{Bun}_H$ is representable and of finite type. The numerical invariant is defined from a line bundle by [HLH23, Lemma 4.8] and hence is standard and satisfies condition (R) by [HL22, Lemma 4.1.15, Lemma 4.1.16].

For condition (S), we show that for any discrete valuation ring $R/\mathbb{C}$ with fraction field $K = \text{Frac}(R)$ and residue field κ and an R -point $x_R \in \text{Bun}_H^{V\mathcal{L}}(R)$, given the Harder–Narasimhan filtration $y_K \in \text{Bun}_{P_\lambda}^{V^{\lambda \geq 0}}(K)$ of the generic fiber x_K , there exists a filtration $y_\kappa \in \text{Bun}_{P_\lambda}^{V^{\lambda \geq 0}}(\kappa)$ of the special fiber x_κ such that

$$\mu(y_\kappa) \geq \mu(y_K).$$

Moreover, when equality holds, the filtration extends to a filtration $y_R \in \text{Bun}_{P_\lambda}^{V^{\lambda \geq 0}}(R)$. To prove this, we first consider the case V is trivial. Using the properness of the Drinfeld compactification, we can extend y_K to a point $y'_R \in \text{Bun}_{P_\lambda}(R)$. Then we can take y_κ to be the saturation of the special fiber $y'_\kappa \in \text{Bun}_{P_\lambda}(\kappa)$. It is clear that this construction satisfies the desired properties. In the general case, since $V^{\lambda \geq 0} \subset V$ is a closed immersion, the condition on the V -section can be checked on the generic fiber, and hence, the same construction works. This proves condition (S).

For condition (B), we follow the strategy in the proof of [HLH23, Proposition 4.6]. That is, we show that for any family $\xi : S \rightarrow \text{Bun}_H^{V\mathcal{L}}$ for a quasi-compact scheme S , there exists a quasi-compact open substack $U \subset \text{Bun}_H^{V\mathcal{L}}$ such that for any geometric point $s \in S(K)$ and a filtration $y \in \text{Bun}_{P_\lambda}^{V^{\lambda \geq 0}}(K)$ of $\xi(s)$, there exists a filtration $y' \in \text{Bun}_{P_{\lambda'}}^{V^{\lambda' \geq 0}}(K)$ of $\xi(s)$ with $\mu(y') \geq \mu(y)$ and $(p_{\lambda'} \circ s_{\lambda'} \circ q_{\lambda'})(y') \in U(K)$. When V is trivial, this is [HLH23, Proposition 4.6]. In general, we use a small variant of the argument. Indeed, we will choose y' to be a coarsening of the filtration y as follows. Take $y \in \text{Bun}_{P_\lambda}^{V^{\lambda \geq 0}}(K)$ as above. If $\mu(y) \leq 0$, we can simply take $\lambda' = 0$ and y' to be the trivial filtration. So we can assume $\mu(y) > 0$. Consider hyperplanes in $X_*(T)_\mathbb{Q}$ defined by the weights of V and roots of $\mathfrak{h}$, and let $\sigma_\lambda \subset X_*(T)_\mathbb{Q}$ be the cone containing λ . Then for any coweight $\lambda' \in \bar{\sigma}_\lambda \setminus \{0\}$, the filtration y induces a filtration $y' \in \text{Bun}_{P_{\lambda'}}^{V^{\lambda' \geq 0}}(K)$ where $P_\lambda \subset P_{\lambda'}$ and $V^{\lambda \geq 0} \subset V^{\lambda' \geq 0}$. We choose $\lambda' \in \bar{\sigma}_\lambda \setminus \{0\}$ to maximize $\mu(y') = (\deg(y'), \lambda') / \|\lambda'\| = (\deg(y), \lambda') / \|\lambda'\|$. The maximizing property implies that we can rescale λ' such that $(\lambda' - \deg(y), \lambda') = (\lambda' - \deg(y'), \lambda') = 0$, where the last equality holds because $\deg(y') \in X_*(Z(L_{\lambda'}))_\mathbb{Q}$ is the orthogonal projection of $\deg(y) \in X_*(Z(L_\lambda))_\mathbb{Q}$ to $X_*(Z(L_{\lambda'}))_\mathbb{Q}$. This implies $\|\lambda'\| = (\lambda', \lambda') / \|\lambda'\| = \mu(y')$. Since S is quasi-compact, there exists a constant C depending only on S , such that $\mu(y'') \leq C$ for any filtration y'' of the underlying H -bundle of $\xi(s)$ for any $s \in S(K)$. This implies that $\|\lambda'\| \leq C$. Moreover, since λ' is the orthogonal projection of $\deg(y)$ on one of finitely many linear subspaces of $X_*(T)_\mathbb{Q}$ (indeed, the finitely many intersections of the hyperplanes introduced above), we know that there exists an integer N which only depends on H, V such that $\lambda' \in \frac{1}{N}X_*(T)$. Therefore, we have $\lambda' \in \frac{1}{N}X_*(T) \cap B(0, C)$, which is a finite set depending only on S ; hence we can treat λ' as fixed. By [HLH23, Lemma 4.10(2)], we only need to show that there exists $d \in X_*(Z(L_{\lambda'}))_\mathbb{Q}$ such that $\deg(y') - d \in X_*(Z(L_{\lambda'}))_\mathbb{Q}^+$. Denote $\delta = \deg(y') - \lambda'$. Since λ' is dominant in $X_*(Z(L_{\lambda'}))_\mathbb{Q}$, we only need to show that there exists a constant C' (depending only on λ', C) such that $\|\delta\| \leq C'$. Since λ' is strictly dominant in $X_*(Z(L_{\lambda'}))_\mathbb{Q}$, there exists a real number $r > 0$ such that $B(\lambda', r) \subset X_*(Z(L_{\lambda'}))_\mathbb{R}^+$. By our choice of the constant C , we know that $(\deg(y'), -) \leq C\|-\|$ holds on $X_*(Z(L_{\lambda'}))_\mathbb{R}^+$. Therefore, we have $(\delta, -) = (\deg(y'), -) - (\lambda', -) \leq (C + \|\lambda'\|)\|-\| \leq 2C\|-\|$ on $X_*(Z(L_{\lambda'}))_\mathbb{R}^+$. In particular, we have $r\|\delta\| = (\delta, \lambda' + r\frac{\delta}{\|\delta\|}) \leq 2C\|\lambda' + r\frac{\delta}{\|\delta\|}\| \leq 2C(C + r)$, which implies $\|\delta\| \leq 2C(C/r + 1)$. We can take $C' = 2C(C/r + 1)$ and conclude the proof. $\square$

3.3. Proof of Lemma 3.5. Consider the map $a : \text{Higgs}_G^M \rightarrow \text{Bun}_H$. Define

$$\mathcal{F} = a_* \mathcal{L}_{H, \text{Std}}^{-1/2} \cong (a_* \mathcal{O}_{\text{Higgs}_G^M}) \otimes \mathcal{L}_{H, \text{Std}}^{-1/2}.$$

We use the Harder–Narasimhan stratification of Bun_H as usual (same as taking V to be the trivial representation in Proposition 3.7). Let $i_\lambda : \text{Bun}_H^{(\lambda)} \rightarrow \text{Bun}_H$ be a Harder–Narasimhan stratum. Let $i_\lambda^\wedge : \text{Bun}_H^{(\lambda), \wedge} \rightarrow$

Bun_H be the formal completion of Bun_H along $\text{Bun}_H^{(\lambda)}$. We have a Cartesian diagram:

$$\begin{array}{ccc} \text{Bun}_H^{(\lambda)} & \longrightarrow & \text{Bun}_{P_\lambda} \\ \downarrow q^{ss} & & \downarrow q \\ \text{Bun}_{L_\lambda}^{\lambda, ss} & \longrightarrow & \text{Bun}_{L_\lambda} \end{array}$$

Let $X_*(T)_{\mathbb{Q}}^{+, bdd} \subset X_*(T)_{\mathbb{Q}}^+$ be the subset of $\lambda \in X_*(T)_{\mathbb{Q}}^+$ such that $\lambda \leq (6g-6)\rho_H$. Then we have $\text{Bun}_H^{bdd} = \bigsqcup_{\lambda \in X_*(T)_{\mathbb{Q}}^{+, bdd}} \text{Bun}_H^{(\lambda)}$. For $\lambda \notin X_*(T)_{\mathbb{Q}}^{+, bdd}$, we want to show that $\Gamma(\text{Bun}_H^{(\lambda), \wedge}, (i_\lambda^\wedge)^! \mathcal{F}) = 0$. For this, we only need to show that

$$\Gamma(\text{Bun}_H^{(\lambda)}, i_\lambda^! \mathcal{F} \otimes \text{Sym}(T_{\text{Bun}_H^{(\lambda)} / \text{Bun}_H} [1])) = 0.$$

Define $N = 6g - 6$. Since $\lambda \notin X_*(T)_{\mathbb{Q}}^{+, bdd}$, there exists a simple root α of H such that $\langle \lambda, \alpha \rangle > N$. Choose $\varpi \in X_*(Z(L_\lambda))_{\mathbb{Q}}$ via the following rule: Let $\varpi_1, \dots, \varpi_r$ be the fundamental coweights of $\mathfrak{h}$ ordered according to the Dynkin diagram such that ϖ_r (or ϖ_{r-1}, ϖ_r) corresponds to the spin node(s). Let $\alpha_1, \dots, \alpha_r$ be the corresponding simple roots. Identifying $X_*(T) \cong \mathbb{Z}^r$ in the standard way, we have $\varpi_{r-1} = (1/2, \dots, 1/2, 1/2, -1/2)$ and $\varpi_r = (1/2, \dots, 1/2, 1/2, 1/2)$ if H is of type D . Let $\lambda = (a_1, \dots, a_{r-1}, a_r)$ with $a_1 \geq \dots \geq a_{r-1} \geq |a_r|$. Using the outer symmetry of the Dynkin diagram of type D , we can assume that $a_r \geq 0$.

- If H is of type B , or H is of type D and $\alpha = \alpha_i \neq \alpha_{r-1}, \alpha_r$, then let $\varpi = \varpi_i$.
- If H is of type D and $a_{r-1} - a_r > N/3$, we take $\varpi = \varpi_{r-1} + \varpi_r$.
- If H is of type D and $a_{r-1} - a_r \leq N/3$. In this case, one has $\alpha = \alpha_r$; therefore, $a_{r-1} + a_r > N$, which implies that $a_r > N/3$. We take $\varpi = 2\varpi_r$.

Consider the action of $\mathbb{B}\mathbb{G}_m$ on Bun_{L_λ} through a multiple of ϖ . We only need to show that $q_*^{ss}(i_\lambda^! \mathcal{F} \otimes \text{Sym}(T_{\text{Bun}_H^{(\lambda)} / \text{Bun}_H} [1]))$ has (strictly) negative weights with respect to this $\mathbb{B}\mathbb{G}_m$ -action.

Note that it admits a filtration with associated graded (up to a cohomological shift)

$$\text{Sym}(\Gamma(C, \mathfrak{h}_{\text{univ}} \otimes \Omega_C)^* \oplus \Gamma(C, \text{Std}_{\text{univ}} \otimes \Omega_C)^* \oplus \Gamma(C, \mathfrak{n}_{\lambda, \text{univ}}^-)[1] \oplus \Gamma(C, \mathfrak{n}_{\lambda, \text{univ}})^*[-1]) \otimes \mathcal{L}_{H, \text{Std}}^{-1/2} \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1}.$$

It remains to show that

$$\text{Sym}(\Gamma(C, \mathfrak{h}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^* \oplus \Gamma(C, \text{Std}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^*) \otimes \mathcal{L}_{H, \text{Std}}^{-1/2} \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1}$$

has strictly negative weights.

By our choice of ϖ , over $\text{Bun}_{L_\lambda}^{\lambda, ss}$, every fiber of $\mathfrak{h}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C$ has slopes less than $-N/3 + (2g-2) = 0$ and every fiber of $\text{Std}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C$ has slopes less than $-N/3 + (2g-2) = 0$. Therefore, the complex above equals

$$\text{Sym}(H^1(C, \mathfrak{h}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^*[1] \oplus H^1(C, \text{Std}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^*[1]) \otimes \mathcal{L}_{H, \text{Std}}^{-1/2} \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1}.$$

Note that

$$\max \text{wt}_\varpi(\text{Sym}(H^1(C, \mathfrak{h}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^*[1]) \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1}) = \text{wt}_\varpi(\mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}) - \text{wt}_\varpi(\mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^-) < 0$$

and

$$\max \text{wt}_\varpi(\text{Sym}(H^1(C, \text{Std}_{\text{univ}}^{\varpi < 0} \otimes \Omega_C)^*[1]) \otimes \mathcal{L}_{H, \text{Std}}^{-1/2}) = (\text{wt}_\varpi(\mathcal{L}_{L_\lambda, \text{Std}^{\varpi > 0}}) - \text{wt}_\varpi(\mathcal{L}_{L_\lambda, \text{Std}^{\varpi < 0}}))/2 < 0,$$

where we use wt_ϖ to denote the weight with respect to the $\mathbb{B}\mathbb{G}_m$ -action and $\max \text{wt}_\varpi$ for the maximal weight. This concludes the proof.

3.4. Proof of Lemma 3.3. Under the identification $\text{Higgs}_G^M \cong \text{Bun}_H^{\mathfrak{h}'_{\Omega_C}}$ where $\mathfrak{h}' = \mathfrak{h} \oplus \text{Std}$, we can use the Harder–Narasimhan stratification of $\text{Bun}_H^{\mathfrak{h}'_{\Omega_C}}$ as in Proposition 3.7. Let $i_\lambda : \text{Bun}_H^{\mathfrak{h}'_{\Omega_C}, (\lambda)} \rightarrow \text{Bun}_H^{\mathfrak{h}'_{\Omega_C}}$ be a

Harder–Narasimhan stratum. We have a Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}, (\lambda)} & \longrightarrow & \mathrm{Bun}_{P_\lambda}^{\mathfrak{h}'_{\Omega_C}^{\lambda \geq 0}} \\ \downarrow q^{\mathrm{ss}} & & \downarrow q \\ \mathrm{Bun}_{L_\lambda}^{\mathfrak{h}'_{\Omega_C}^{\lambda=0}, \lambda, \mathrm{ss}} & \longrightarrow & \mathrm{Bun}_{L_\lambda}^{\mathfrak{h}'_{\Omega_C}^{\lambda=0}} \end{array}$$

We proceed as in the proof of Lemma 3.5. We only need to show that for any nonzero $\lambda \in X_*(T)_{\mathbb{Q}}^+$

$$\Gamma(\mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}, (\lambda)}, i_\lambda^! \mathcal{L}_{H, \mathrm{Std}}^{-1/2} \otimes \mathrm{Sym}(T_{\mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}, (\lambda)} / \mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}} [1])) = 0.$$

Consider the action of $\mathbb{B}\mathbb{G}_m$ on $\mathrm{Bun}_{L_\lambda}^{\mathfrak{h}'_{\Omega_C}^{\lambda=0}}$ through a multiple of λ . We only need to show that

$$q_*^{\mathrm{ss}}(i_\lambda^! \mathcal{L}_{H, \mathrm{Std}}^{-1/2} \otimes \mathrm{Sym}(T_{\mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}, (\lambda)} / \mathrm{Bun}_H^{\mathfrak{h}'_{\Omega_C}} [1]))$$

has strictly negative weights with respect to this $\mathbb{B}\mathbb{G}_m$ -action. This complex has a filtration with associated graded (up to a cohomological shift)

$$\mathrm{Sym}(\Gamma(C, [\mathfrak{n}_{\lambda, \mathrm{univ}}^- \rightarrow (\mathfrak{h}'_{\mathrm{univ}})^{\lambda < 0} \otimes \Omega_C]) \oplus \Gamma(C, [\mathfrak{n}_{\lambda, \mathrm{univ}} \rightarrow (\mathfrak{h}'_{\mathrm{univ}})^{\lambda > 0} \otimes \Omega_C])^*) \otimes \mathcal{L}_{H, \mathrm{Std}}^{-1/2} \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1} \otimes \mathcal{L}_{L_\lambda, \mathfrak{h}'_{\Omega_C}^{\lambda < 0}}$$

where both $[\mathfrak{n}_{\lambda, \mathrm{univ}}^- \rightarrow (\mathfrak{h}'_{\mathrm{univ}})^{\lambda < 0} \otimes \Omega_C]$ and $[\mathfrak{n}_{\lambda, \mathrm{univ}} \rightarrow (\mathfrak{h}'_{\mathrm{univ}})^{\lambda > 0} \otimes \Omega_C]$ are placed in degree $-1, 0$. The desired vanishing follows because the maximal weight is

$$\mathrm{wt}_\lambda(\mathcal{L}_{H, \mathrm{Std}}^{-1/2} \otimes \mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^{-1} \otimes \mathcal{L}_{L_\lambda, \mathfrak{h}'_{\Omega_C}^{\lambda < 0}}) = (\mathrm{wt}_\lambda(\mathcal{L}_{L_\lambda, \mathrm{Std}^{\lambda > 0}}) - \mathrm{wt}_\lambda(\mathcal{L}_{L_\lambda, \mathrm{Std}^{\lambda < 0}}))/2 + (\mathrm{wt}_\lambda(\mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}) - \mathrm{wt}_\lambda(\mathcal{L}_{L_\lambda, \mathfrak{n}_\lambda}^-)) < 0.$$

This concludes the proof.

4. PERIOD OF DIFFERENTIAL OPERATORS

In this section, we compute the $*$ -period integral of $D_{\mathrm{Bun}_G, \kappa} \in D\text{-mod}(\mathrm{Bun}_G)$. More precisely, let $\mathrm{Op}_{\tilde{G}}$ be the moduli of $\tilde{G}$ -opers on C and $p_{\mathrm{Op}} : \mathrm{Op}_{\tilde{G}} \rightarrow \mathrm{Loc}_{\tilde{G}}$ be the natural map.⁸ We will prove the following:

Proposition 4.1. *There exists $\sigma \in \mathrm{Op}_{\tilde{G}}(C)$ such that the fiber $i_\sigma^*(\mathbb{L}_G(\mathcal{P}_X))[\delta_H - \delta_{N_{\Omega_C^{1/2}}}] \in \mathrm{Vect}$ lies in $\mathrm{Vect}^\heartsuit$ and has dimension $2^{(g-1)\dim(\tilde{M})}$, where $i_\sigma : \{\sigma\} \rightarrow \mathrm{Loc}_{\tilde{G}}$ is the induced inclusion.*

We prove Proposition 4.1 by passing to the classical limit, where the corresponding statement is Corollary 3.6.

4.1. Filtered D -modules. We first recall the theory of filtered D -modules on smooth algebraic stacks following [FR25, §6.2].

4.1.1. Hodge stack. For a smooth algebraic stack $\mathcal{Y}$, there is a *Hodge stack* $\mathcal{Y}_{h, \mathrm{dR}} \rightarrow [\mathbb{A}_h^1 / \mathbb{G}_m]$. Its fiber over $\hbar = 1$ is $\mathcal{Y}_{\mathrm{dR}}$, and its fiber over $\hbar = 0$ is $\mathbb{B}_\mathcal{Y} T_\mathcal{Y}^\wedge$. The category of filtered D -modules is defined by

$$D\text{-mod}_\hbar(\mathcal{Y}) := \mathrm{IndCoh}(\mathcal{Y}_{h, \mathrm{dR}}).$$

We have natural maps $j_1 : \mathcal{Y}_{\mathrm{dR}} \rightarrow \mathcal{Y}_{h, \mathrm{dR}}$ and $i_0 : \mathbb{B}_\mathcal{Y} T_\mathcal{Y}^\wedge \rightarrow \mathcal{Y}_{h, \mathrm{dR}}$. We have the forgetful functor

$$\mathrm{oblv} := j_1^! : D\text{-mod}_\hbar(\mathcal{Y}) \cong \mathrm{IndCoh}(\mathcal{Y}_{h, \mathrm{dR}}) \rightarrow D\text{-mod}(\mathcal{Y}) \cong \mathrm{IndCoh}(\mathcal{Y}_{\mathrm{dR}}).$$

Koszul duality gives $\mathrm{IndCoh}(\mathbb{B}_\mathcal{Y} T_\mathcal{Y}^\wedge) \cong \mathrm{QCoh}(T^*\mathcal{Y})$. We have the functor taking associated graded

$$\mathrm{gr} := i_0^! : D\text{-mod}_\hbar(\mathcal{Y}) \cong \mathrm{IndCoh}(\mathcal{Y}_{h, \mathrm{dR}}) \rightarrow \mathrm{QCoh}(T^*\mathcal{Y}) \cong \mathrm{IndCoh}(\mathbb{B}_\mathcal{Y} T_\mathcal{Y}^\wedge).$$

⁸We use the standard convention for opers as in [Gai15, §10.2] which is different from the one used in [BD91] when $\tilde{G}$ is not of adjoint type.

4.1.2. *Functoriality.* Given a morphism between smooth algebraic stacks $f : \mathcal{Y} \rightarrow \mathcal{Z}$, we have the induced correspondence on cotangent stacks

$$\begin{array}{ccc} T^*\mathcal{Z} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{Pf} & T^*\mathcal{Z} \\ \downarrow Df & & \\ T^*\mathcal{Y} & & \end{array} .$$

We use

$$f_{\mathrm{dR},*,\mathrm{ren}} : \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Z})$$

to denote the renormalized direct image functor defined by continuously extending the functor

$$f_{\mathrm{dR},*} : \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Z})$$

on compact objects. The functors

$$f_{\mathrm{dR}}^! : \mathrm{D}\text{-mod}(\mathcal{Z}) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Y})$$

and

$$f_{\mathrm{dR},*,\mathrm{ren}} : \mathrm{D}\text{-mod}(\mathcal{Y}) \rightarrow \mathrm{D}\text{-mod}(\mathcal{Z})$$

admit natural filtered enhancements, for which we use the same notation. By [FR25, (6.2.1)], one has commutative diagrams of functors:

$$\begin{array}{ccc} \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Z}) & \xrightarrow{f_{\mathrm{dR}}^!} & \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Y}) \\ \downarrow \mathrm{gr} & & \downarrow \mathrm{gr} \\ \mathrm{QCoh}(T^*\mathcal{Z}) & \xrightarrow{(Df)_*(Pf)^!} & \mathrm{QCoh}(T^*\mathcal{Y}) \\ \\ \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Y}) & \xrightarrow{f_{\mathrm{dR},*,\mathrm{ren}}} & \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Z}) \\ \downarrow \mathrm{gr} & & \downarrow \mathrm{gr} \\ \mathrm{QCoh}(T^*\mathcal{Y}) & \xrightarrow{(Pf)^*(Df)^*} & \mathrm{QCoh}(T^*\mathcal{Z}) \end{array} .$$

4.1.3. *Filtration on differential operators.* Let $\mathcal{Y}$ be a smooth algebraic stack. There is a natural map $q_{\mathcal{Y},\hbar} : \mathcal{Y} \times [\mathbb{A}_{\hbar}^1/\mathbb{G}_m] \rightarrow \mathcal{Y}_{\hbar,\mathrm{dR}}$. This induces an adjoint pair

$$\mathrm{ind}_{\hbar}^r := (q_{\mathcal{Y},\hbar})_* : \mathrm{IndCoh}(\mathcal{Y} \times [\mathbb{A}_{\hbar}^1/\mathbb{G}_m]) \rightleftarrows \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Y}) : \mathrm{oblv}_{\hbar}^r := (q_{\mathcal{Y},\hbar})^!.$$

In particular, we have the filtered D -module $\mathrm{D}_{\hbar,\mathcal{Y}} := \mathrm{ind}_{\hbar}^r(\omega_{\mathcal{Y} \times [\mathbb{A}_{\hbar}^1/\mathbb{G}_m]}) \in \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Y})$ satisfying

$$\mathrm{oblv}(\mathrm{D}_{\hbar,\mathcal{Y}}) \cong \mathrm{D}_{\mathcal{Y}} \in \mathrm{D}\text{-mod}(\mathcal{Y})$$

and

$$\mathrm{gr}(\mathrm{D}_{\hbar,\mathcal{Y}}) \cong \pi_{\mathcal{Y}}^* \omega_{\mathcal{Y}}$$

where $\pi_{\mathcal{Y}} : T^*\mathcal{Y} \rightarrow \mathcal{Y}$ is the projection. More generally, for any line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{Y})$, one can define the category of twisted filtered D -modules $\mathrm{D}\text{-mod}_{\hbar,\mathcal{L}}(\mathcal{Y})$ and the object $\mathrm{D}_{\hbar,\mathcal{Y},\mathcal{L}} \in \mathrm{D}\text{-mod}_{\hbar,\mathcal{L}}(\mathcal{Y})$ via the standard twisting construction. Using the canonical equivalence $\mathrm{D}\text{-mod}_{\hbar,\mathcal{L}}(\mathcal{Y}) \cong \mathrm{D}\text{-mod}_{\hbar}(\mathcal{Y})$, we have

$$\mathrm{oblv}(\mathrm{D}_{\hbar,\mathcal{Y},\mathcal{L}}) \cong \mathrm{D}_{\mathcal{Y},\mathcal{L}} \in \mathrm{D}\text{-mod}_{\mathcal{L}}(\mathcal{Y})$$

and

$$\mathrm{gr}(\mathrm{D}_{\hbar,\mathcal{Y},\mathcal{L}}) \cong \pi_{\mathcal{Y}}^*(\omega_{\mathcal{Y}} \otimes \mathcal{L}^{-1}).$$

4.2. Quantization of the Hitchin system. In this section, we review the result of [BD91] needed for the computation.

The following proposition follows from [ABC⁺24a, Theorem 18.5.2].⁹

Proposition 4.2. *There is an isomorphism $\mathbb{L}_G(\mathrm{D}_{\mathrm{Bun}_G, \kappa}) \cong (p_{\mathrm{Op}})_* \mathcal{O}_{\mathrm{Op}_{\check{G}}}[\delta_{N_{\Omega_G^{1/2}}}] \in \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{\check{G}})$.*

In particular, there is a natural map

$$\mathcal{O}(\mathrm{Op}_{\check{G}}) \rightarrow \mathrm{End}(\mathrm{D}_{\mathrm{Bun}_G, \kappa}).$$

Recall that we have the Hitchin map $h_G : \mathrm{Higgs}_G \rightarrow \mathrm{Hitch}_G$. The main result of [BD91] says that $\mathcal{O}(\mathrm{Op}_{\check{G}}) \rightarrow \mathrm{End}(\mathrm{D}_{\mathrm{Bun}_G, \kappa})$ is a quantization of the map

$$\mathcal{O}(\mathrm{Hitch}_G) \rightarrow \mathrm{End}((\mathcal{L}_{G, \mathfrak{g}}^{-1/2} \otimes \omega_{\mathrm{Bun}_G})|_{\mathrm{Higgs}_G})$$

induced by h_G .

More precisely, let $\mathrm{Op}_{\check{G}, \hbar}$ be the moduli of $\hbar$ -opers for $\check{G}$ (replacing connections by $\hbar$ -connections), which is a family over $[\mathbb{A}_{\hbar}^1/\mathbb{G}_m]$, whose fiber at $\hbar = 0$ is Hitch_G and the fiber at $\hbar = 1$ is $\mathrm{Op}_{\check{G}}$. Let $\mathrm{D}_{\hbar, \mathrm{Bun}_G, \kappa} \in \mathrm{D}\text{-mod}_{\hbar}(\mathrm{Bun}_G)$ be the natural filtered D -module on Bun_G , whose fiber at $\hbar = 0$ is $(\mathcal{L}_{G, \mathfrak{g}}^{-1/2} \otimes \omega_{\mathrm{Bun}_G})|_{\mathrm{Higgs}_G} \in \mathrm{QCoh}(\mathrm{Higgs}_G)$ and the fiber at $\hbar = 1$ is $\mathrm{D}_{\mathrm{Bun}_G, \kappa} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$. The following is a reformulation of the main result of [BD91].

Proposition 4.3. *There is a natural map of algebra objects $\mathcal{O}(\mathrm{Op}_{\check{G}, \hbar}) \rightarrow \underline{\mathrm{End}}_{\mathrm{QCoh}([\mathbb{A}_{\hbar}^1/\mathbb{G}_m])}(\mathrm{D}_{\hbar, \mathrm{Bun}_G, \kappa})$, whose fibers at $\hbar = 0$ and $\hbar = 1$ are the maps considered above.*

4.3. Computation of the period of differential operators. In this section, we compute the $*$ -period integral of $\mathrm{D}_{\mathrm{Bun}_G, \kappa} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$. More precisely, we will prove the following result.

Proposition 4.4. *The object $\mathrm{Hom}(\mathcal{P}_X, \mathrm{D}_{\mathrm{Bun}_G, \kappa}) \in \mathrm{Mod}_{\mathcal{O}(\mathrm{Op}_{\check{G}})}$ admits an increasing filtration that is bounded below such that the associated graded is isomorphic to $\Gamma(\mathrm{Higgs}_G^{M, \mathrm{bdd}}, \mathcal{L}_{H, \mathrm{Std}}^{-1/2}[\delta_H]) \in \mathrm{Mod}_{\mathcal{O}(\mathrm{Hitch}_G)}$.*

Proof. We start with the following lemma. Recall $j_{\mathrm{cusp}} : \mathrm{Bun}_G^{\mathrm{cusp}} \rightarrow \mathrm{Bun}_G$ defined in (2.1).

Lemma 4.5. *The object $\mathrm{D}_{\mathrm{Bun}_G, \kappa} \in \mathrm{D}\text{-mod}(\mathrm{Bun}_G)$ is cuspidal. In particular, one has*

$$\mathrm{D}_{\mathrm{Bun}_G, \kappa} \simeq (j_{\mathrm{cusp}})_* \mathrm{D}_{\mathrm{Bun}_G^{\mathrm{cusp}}, \kappa}.$$

Proof. Since the map $p_{\mathrm{Op}} : \mathrm{Op}_{\check{G}} \rightarrow \mathrm{Loc}_{\check{G}}$ factors through $\mathrm{Loc}_{\check{G}}^{\mathrm{irr}} \subset \mathrm{Loc}_{\check{G}}$, the statement follows from [FR25, Corollary 11.2.2.3]. $\square$

Recall $j_{\mathrm{bdd}} : \mathrm{Bun}_H^{\mathrm{bdd}} \rightarrow \mathrm{Bun}_H$ defined in (3.1). The restriction of $\pi_X : \mathrm{Bun}_H \rightarrow \mathrm{Bun}_G$ induces a map $\pi_X^{\mathrm{bdd}} : \mathrm{Bun}_H^{\mathrm{bdd}} \rightarrow \mathrm{Bun}_G$ which satisfies $\pi_X^{-1}(\mathrm{Bun}_G^{\mathrm{cusp}}) \subset \mathrm{Bun}_H^{\mathrm{bdd}}$. By Lemma 4.5, we have

$$\begin{aligned} & \mathrm{Hom}(\mathcal{P}_X, \mathrm{D}_{\mathrm{Bun}_G, \kappa}) \\ &= \Gamma_{\mathrm{dR}}(\mathrm{Bun}_G^X, \pi_X^! \mathrm{D}_{\mathrm{Bun}_G, \kappa}) \\ &= \Gamma_{\mathrm{dR}}(\mathrm{Bun}_H^{\mathrm{bdd}}, (\pi_X^{\mathrm{bdd}})^! \mathrm{D}_{\mathrm{Bun}_G, \kappa}) \end{aligned} \quad (4.1)$$

We note the following lemma.

Lemma 4.6. *We have $\Gamma_{\mathrm{dR}, \mathrm{ren}}(\mathrm{Bun}_H^{\mathrm{bdd}}, (\pi_X^{\mathrm{bdd}})^! \mathrm{D}_{\mathrm{Bun}_G, \kappa}) \cong \Gamma_{\mathrm{dR}}(\mathrm{Bun}_H^{\mathrm{bdd}}, (\pi_X^{\mathrm{bdd}})^! \mathrm{D}_{\mathrm{Bun}_G, \kappa})$.*

Proof. We note the following general fact: Let $\mathcal{Y}$ be a smooth QCA stack. Then $\mathrm{QCoh}(\mathcal{Y})$ is compactly generated and $\Gamma(\mathcal{Y}, -) : \mathrm{QCoh}(\mathcal{Y}) \rightarrow \mathrm{Vect}$ is continuous. Therefore, we have

$$\Gamma_{\mathrm{dR}, \mathrm{ren}}(\mathcal{Y}, -) \circ \mathrm{ind}^l \cong \Gamma(\mathcal{Y}, - \otimes \omega_{\mathcal{Y}}) \cong \Gamma_{\mathrm{dR}}(\mathcal{Y}, -) \circ \mathrm{ind}^l.$$

Since $\mathrm{Bun}_H^{\mathrm{bdd}}$ is QCA, we only need to show that

$$(\pi_X^{\mathrm{bdd}})^! \mathrm{D}_{\mathrm{Bun}_G, \kappa} \in \mathrm{ind}^l(\mathrm{QCoh}(\mathrm{Bun}_H^{\mathrm{bdd}})) \subset \mathrm{D}\text{-mod}(\mathrm{Bun}_H^{\mathrm{bdd}}).$$

⁹We ignore some line twists in [ABC⁺24a, Theorem 18.5.2] which are not relevant for our purpose.

Let $f_{\text{bdd}} : \text{Bun}_H^{\text{bdd}} \rightarrow \text{Bun}_{H'}$ be the natural map. We have

$$\begin{aligned} & (\pi_X^{\text{bdd}})^! D_{\text{Bun}_G, \kappa} \\ & \cong (\pi_X^{\text{bdd}})^! \text{ind}^l(\mathcal{L}_{G, \mathfrak{g}}^{-1/2}) \\ & \cong \text{ind}^l(\mathcal{L}_{H, \mathfrak{h}}^{-1/2}|_{\text{Bun}_H^{\text{bdd}}}) \otimes^! f_{\text{bdd}}^! \text{ind}^l(\mathcal{L}_{H', \mathfrak{h}'}^{-1/2}) \\ & \cong \text{ind}^l(\mathcal{L}_{H, \mathfrak{h}}^{-1/2}|_{\text{Bun}_H^{\text{bdd}}}) \otimes \text{oblv}^l f_{\text{bdd}}^! \text{ind}^l(\mathcal{L}_{H', \mathfrak{h}'}^{-1/2}) \end{aligned}$$

This concludes the proof. $\square$

By the discussion in §4.1.2, we obtain the natural increasing filtration on $\Gamma_{\text{dR}, \text{ren}}(\text{Bun}_H^{\text{bdd}}, (\pi_X^{\text{bdd}})^! D_{\text{Bun}_G, \kappa})$. By (4.1) and Lemma 4.6, we obtain the natural increasing filtration on $\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa})$ compatible with the filtration on $\mathcal{O}(\text{Op}_{\tilde{G}})$. The filtration is bounded below by construction. Consider the diagram of cotangent stacks

$$\begin{array}{ccc} \text{Higgs}_G^{M, \text{bdd}} & \longrightarrow & T^* \text{Bun}_G \times_{\text{Bun}_G} \text{Bun}_H^{\text{bdd}} \xrightarrow{P\pi_X^{\text{bdd}}} T^* \text{Bun}_G \\ \downarrow & & \downarrow D\pi_X^{\text{bdd}} \\ \text{Bun}_H^{\text{bdd}} & \xrightarrow{Dp_X} & T^* \text{Bun}_H^{\text{bdd}} \\ \downarrow Pp_X & & \\ * & & \end{array}$$

where the square is Cartesian and $p_X : \text{Bun}_H^{\text{bdd}} \rightarrow *$ is the map to a point. By the discussion in §4.1.2 and §4.1.3, we have an isomorphism

$$\begin{aligned} & \text{gr}(\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa})) \\ & \cong \Gamma(\text{Bun}_H^{\text{bdd}}, (Dp_X)^*(D\pi_X^{\text{bdd}})_*(P\pi_X^{\text{bdd}})^!(\mathcal{L}_{G, \mathfrak{g}}^{-1/2} \otimes \omega_{\text{Bun}_G})) \\ & \cong \Gamma(\text{Higgs}_G^{M, \text{bdd}}, \mathcal{L}_{G, \mathfrak{g}}^{-1/2} \otimes \omega_{\text{Bun}_H^{\text{bdd}}}) \\ & \cong \Gamma(\text{Higgs}_G^{M, \text{bdd}}, \mathcal{L}_{H, \text{Std}}^{-1/2})[\delta_H] \end{aligned}$$

which is compatible with the $\mathcal{O}(\text{Hitch}_G)$ -module structure by Proposition 4.3. This concludes the proof. $\square$

Remark 4.7. Our proof of Proposition 4.4 can be compared with the proof of [FR25, Theorem 6.1.2.1] for the Whittaker period. Both proofs pass to the classical limit, but the technical challenges are different. In our setting, the filtration is automatically bounded below, unlike in [FR25], where the microlocal geometry is non-polarized. On the other hand, we need to replace Bun_H with a quasi-compact stack $\text{Bun}_H^{\text{bdd}}$ and establish Lemma 4.6 to ensure the exhaustiveness of the filtration, whereas exhaustiveness is automatic in [FR25]. Moreover, the classical limit is considerably more complicated in our setting. Indeed, the relative Hitchin fibration is much more involved than in the Whittaker case, where the corresponding Higgs_G^M is the Kostant section. This leads to the additional calculations in §3.

4.4. Proof of Proposition 4.1. The following lemma is standard:

Lemma 4.8. *Let $A \in \text{Alg}(\text{QCoh}([\mathbb{A}_h^1/\mathbb{G}_m]))$ be a filtered algebra, and let $M \in \text{Mod}_A$ be a filtered A -module. Assume that:*

- (1) *The filtrations on A and M are increasing and bounded below.*
- (2) *$A/\hbar A \in \text{Vect}^\heartsuit$.*
- (3) *The associated graded $M/\hbar M \in \text{Mod}_{A/\hbar A}$ is perfect.*

Then M is perfect as an A -module.

Proof. Note that $A/\hbar A \in \text{Vect}^\heartsuit$ implies that $H^i(A)/\hbar H^i(A) = 0$ for $i \neq 0$, which implies that $H^i(A) = 0$ for $i \neq 0$. Therefore, we have $A \in \text{Vect}^\heartsuit$. Assume $M/\hbar M$ has Tor amplitude $[a, b]$. By the same argument we know that $M \in \text{Mod}_A^{\leq b}$. Take a finitely generated projective module $\overline{P} \in \text{Mod}_{A/\hbar A}^\heartsuit$ and a map

$\overline{P}[-b] \rightarrow M/\hbar M$ inducing $\overline{P} \rightarrow H^b(M/\hbar M)$ (when $a = b$, we simply take $\overline{P} = (M/\hbar M)[b]$). We can find a finitely generated projective module $P \in \text{Mod}_A^\heartsuit$ and a lift $P[-b] \rightarrow M$. Then $\text{Cone}(P[-b] \rightarrow M)/\hbar$ has Tor amplitude $[a, b-1]$. Iterating this process shows that M is perfect. $\square$

By Lemma 4.8 and Corollary 3.6, we know that $\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa}) \in \text{Mod}_{\mathcal{O}(\text{Op}_G)}$ is perfect. Moreover, there exists $\sigma \in \text{Op}_G(\mathbb{C})$ such that $i_\sigma^*(\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa}))[-\delta_H] \in \text{Vect}^\heartsuit$ and has dimension $2^{(g-1)\dim(M)}$. We have

$$\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa}) \cong \text{Hom}(\mathbb{L}_G(\mathcal{P}_X), (p_{\text{Op}})_* \mathcal{O}_{\text{Op}_G}[\delta_{N_{\Omega^{1/2}}_C}]) \cong (p_{\text{Op}}^* \mathbb{L}_G(\mathcal{P}_X))^\vee[\delta_{N_{\Omega^{1/2}}_C}] \in \text{QCoh}(\text{Op}_G)$$

where $(p_{\text{Op}}^* \mathbb{L}_G(\mathcal{P}_X))^\vee$ is the linear dual of $p_{\text{Op}}^* \mathbb{L}_G(\mathcal{P}_X) \in \text{QCoh}(\text{Op}_G)$ which is well-defined by Lemma 2.2. This implies that

$$i_\sigma^*(\mathbb{L}_G(\mathcal{P}_X))[\delta_H - \delta_{N_{\Omega^{1/2}}_C}] \cong i_\sigma^*(\text{Hom}(\mathcal{P}_X, D_{\text{Bun}_G, \kappa}))^\vee[\delta_H] \in \text{Vect}^\heartsuit$$

and has dimension $2^{(g-1)\dim(M)}$. This concludes the proof of Proposition 4.1.

REFERENCES

- [ABC⁺24a] Dima Arinkin, Dario Beraldo, Justin Campbell, Lin Chen, Joakim Faergeman, Dennis Gaitsgory, Kevin Lin, Sam Raskin, and Nick Rozenblyum, *Proof of the geometric Langlands conjecture II: Kac-Moody localization and the FLE*, arXiv preprint arXiv:2405.03648 (2024). [2](#), [6](#), [7](#), [8](#), [18](#)
- [ABC⁺24b] Dima Arinkin, Dario Beraldo, Lin Chen, Joakim Faergeman, Dennis Gaitsgory, Kevin Lin, Sam Raskin, and Nick Rozenblyum, *Proof of the geometric Langlands conjecture IV: ambidexterity*, arXiv preprint arXiv:2409.08670 (2024). [2](#)
- [Alp13] Jarod Alper, *Good moduli spaces for Artin stacks*, Annales de l'Institut Fourier, vol. 63, 2013, pp. 2349–2402. [12](#), [13](#), [14](#)
- [BD91] Alexander Beilinson and Vladimir Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, 1991. [3](#), [16](#), [18](#)
- [BF08] Roman Bezrukavnikov and Michael Finkelberg, *Equivariant Satake category and Kostant-Whittaker reduction*, 2008. [6](#)
- [BFT22] Alexander Braverman, Michael Finkelberg, and Roman Travkin, *Orthosymplectic Satake equivalence*, Communications in Number Theory and Physics **16** (2022), no. 4, 695–732. [3](#), [9](#), [10](#)
- [BZSV24] David Ben-Zvi, Yiannis Sakellaridis, and Akshay Venkatesh, *Relative Langlands duality*, 2024. [1](#), [2](#), [9](#), [10](#)
- [CCF⁺24] Justin Campbell, Lin Chen, Joakim Faergeman, Dennis Gaitsgory, Kevin Lin, Sam Raskin, and Nick Rozenblyum, *Proof of the geometric Langlands conjecture III: compatibility with parabolic induction*, arXiv preprint arXiv:2409.07051 (2024). [2](#)
- [DG16] Vladimir Drinfeld and Dennis Gaitsgory, *Geometric constant term functor(s)*, Selecta Mathematica **22** (2016), no. 4, 1881–1951. [4](#)
- [FR25] Joakim Faergeman and Sam Raskin, *Non-vanishing of geometric Whittaker coefficients for reductive groups*, J. Amer. Math. Soc. **38** (2025), no. 4, 919–995. [3](#), [16](#), [17](#), [18](#), [19](#)
- [Gai15] Dennis Gaitsgory, *Outline of the proof of the geometric Langlands conjecture for GL_2* , Astérisque (2015), no. 370, 1–112. MR 3364744 [16](#)
- [GR20] Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry: Volume II: Deformations, Lie theory and formal geometry*, vol. 221, American Mathematical Society, 2020. [4](#), [5](#)
- [GR24a] Dennis Gaitsgory and Sam Raskin, *Proof of the geometric Langlands conjecture I: construction of the functor*, arXiv preprint arXiv:2405.03599 (2024). [2](#), [4](#)
- [GR24b] ———, *Proof of the geometric Langlands conjecture V: The multiplicity one theorem*, arXiv preprint arXiv:2409.09856 (2024). [2](#)
- [GR25] ———, *Geometric Langlands in positive characteristic from characteristic zero*, 2025. [2](#)
- [HL22] Daniel Halpern-Leistner, *On the structure of instability in moduli theory*, 2022. [13](#), [14](#)
- [HLH23] Daniel Halpern-Leistner and Andres Fernandez Herrero, *The structure of the moduli of gauged maps from a smooth curve*, 2023. [12](#), [13](#), [14](#)
- [HLM24] Thomas Hameister, Zhilin Luo, and Benedict Morrissey, *Relative Dolbeault geometric Langlands via the regular quotient*, arXiv preprint arXiv:2409.15691 (2024). [12](#)
- [LW25] Shurui Liu and Zeyu Wang, *Higher period integrals and derivatives of L-functions*, 2025. [2](#), [9](#), [10](#), [11](#)
- [Roz21] Nick Rozenblyum, *Connections on moduli spaces and infinitesimal Hecke modifications*, 2021. [8](#), [9](#)
- [Sch08] Alexander H.W. Schmitt, *Geometric invariant theory and decorated principal bundles*, vol. 11, European Mathematical Society, 2008. [13](#), [14](#)
- [Wan25] Zeyu Wang, *Special cycle on Shtukas and categorical trace*, 2025. [2](#)
- [Wan26] ———, *Diagonal cycles on Shtukas and the adjoint L-function*, 2026. [2](#)
- [YZ17] Zhiwei Yun and Wei Zhang, *Shtukas and the Taylor expansion of L-functions*, Annals of Mathematics **186** (2017), no. 3, 767 – 911. [2](#)

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA

Email address: `kychen03@mit.edu`

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, DEPARTMENT OF MATHEMATICS, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139, USA

Email address: `wangzeyu@mit.edu`